

Random trees constructed by aggregation

Nicolas Curien* & Bénédicte Haas†

Abstract

We study a general procedure that builds random \mathbb{R} -trees by gluing recursively a new branch on a uniform point of the pre-existing tree. The aim of this paper is to see how the asymptotic behavior of the sequence of lengths of branches influences some geometric properties of the limiting tree, such as compactness and Hausdorff dimension. In particular, when the sequence of lengths of branches behaves roughly like $n^{-\alpha}$ for some $\alpha \in (0, 1]$, we show that the limiting tree is a compact random tree of Hausdorff dimension α^{-1} . This encompasses the famous construction of the Brownian tree of Aldous. When $\alpha > 1$, the limiting tree is thinner and its Hausdorff dimension is always 1. In that case, we show that α^{-1} corresponds to the dimension of the set of leaves of the tree.

Introduction

Consider a sequence of closed segments or “branches” of lengths $a_1, a_2, a_3, \dots > 0$ and let

$$A_i = a_1 + \dots + a_i, \quad i \geq 1$$

denote the partial sums of their lengths. We construct a sequence of random trees $(\mathcal{T}_n)_{n \geq 1}$ by starting with the tree \mathcal{T}_1 made of the single branch of length a_1 and then recursively gluing the branch of length a_i on a point uniformly distributed (for the length measure) on \mathcal{T}_{i-1} . Let \mathcal{T} be the completion of the increasing union of the \mathcal{T}_n which is thus a random complete continuous tree. Our first result shows that even if the series $\sum a_i$ is divergent, provided that the sequence $\mathbf{a} = (a_i)_{i \geq 1}$ is sufficiently well-behaved, the tree \mathcal{T} is a compact random tree with a fractal behavior.

Theorem 1 (Case $\alpha \leq 1$). *Suppose that there exists $\alpha \in (0, 1]$ such that*

$$a_i \leq i^{-\alpha+o(1)} \quad \text{and} \quad A_i = i^{1-\alpha+o(1)} \quad \text{as } i \rightarrow \infty.$$

Then \mathcal{T} is almost surely a compact real tree of Hausdorff dimension α^{-1} .

We actually get more complete results. On the one hand, the tree \mathcal{T} is compact and has a Hausdorff dimension smaller than α^{-1} as soon as $a_i \leq i^{-\alpha+o(1)}$ for some $\alpha \in (0, 1]$ (Proposition 9). On the other hand, its Hausdorff dimension is larger than α^{-1} as soon as $A_i \geq i^{1-\alpha+o(1)}$ for some $\alpha \in (0, 1]$ (Proposition 12 – this result actually holds under a mild additional assumption that will be discussed in the core of the paper). Let us also mention that in a recent paper [2], Amini et al. considered the same aggregation model and obtained a necessary and sufficient condition for \mathcal{T} to be bounded in the particular case when \mathbf{a} is decreasing, see the discussion in Section 1.4.

Theorem 1 encompasses the famous line-breaking construction of the Brownian continuum random tree (CRT) of Aldous. Specifically, if the sequence \mathbf{a} is the random sequence of lengths given by the

*Université Paris-Sud, E-mail: nicolas.curien@gmail.com

†Université Paris-Dauphine, E-mail: haas@ceremade.dauphine.fr

intervals in a Poisson process on \mathbb{R}_+ with intensity $t dt$, then Aldous proved [1] that \mathcal{T} is compact and of Hausdorff dimension 2 (this was the initial definition of the Brownian CRT). Yet, it is a simple exercise to see that such sequences almost surely satisfy the assumptions of our theorem for $\alpha = 1/2$. More generally, random trees built from a sequence of branches given by the intervals of a Poisson process of intensity $t^\beta dt$ on \mathbb{R}_+ with $\beta > 0$ satisfy our assumptions with $\alpha = \beta/(\beta + 1)$. Typically, in these examples, the sequence \mathbf{a} is not monotonic. See also [6] for a recent construction of the so-called stable trees *via* a similar aggregation procedure which however does not fall in our setup.

When the series $\sum a_i$ is convergent the situation may seem easier. In such cases, it should be intuitive that the limiting tree is compact and of Hausdorff dimension 1. We will see that this is true regardless of the mechanism used to glue the branches together (Proposition 15). But we can go further: when the asymptotic behavior of the sequence \mathbf{a} is sufficiently regular, the set of leaves of \mathcal{T} exhibits an interesting fractal behavior similar to Theorem 1. We recall that the leaves of a continuous tree \mathcal{T} are the points x such that $\mathcal{T} \setminus \{x\}$ stays connected.

Theorem 2 (Case $\alpha > 1$). *Suppose that there exists $\alpha > 1$ such that*

$$a_i \leq i^{-\alpha+o(1)} \quad \text{and} \quad a_i + a_{i+1} + \dots + a_{2i} = i^{1-\alpha+o(1)} \quad \text{as } i \rightarrow \infty.$$

Then the set of leaves of \mathcal{T} is almost surely of Hausdorff dimension α^{-1} .

We can decompose the tree \mathcal{T} into its set of leaves $\text{Leaves}(\mathcal{T})$ and its skeleton $\mathcal{T} \setminus \text{Leaves}(\mathcal{T})$. Since the skeleton is a countable union of segments, its Hausdorff dimension is 1 and so $\dim_{\text{H}}(\mathcal{T}) = 1 \vee \dim_{\text{H}}(\text{Leaves}(\mathcal{T}))$. Theorem 1 and Theorem 2 thus imply that when $a_i = i^{-\alpha}$ for some $\alpha \in (0, \infty)$, the tree \mathcal{T} is compact and

$$\dim_{\text{H}}(\text{Leaves}(\mathcal{T})) = \alpha^{-1}$$

almost surely. When $\alpha = 1$, the Hausdorff dimension of the leaves of \mathcal{T} is not explicitly given in these theorems, but will be calculated further in the text.

A TOY-MODEL FOR DLA. Apart from the abundant random tree literature and the initial definition of the Brownian CRT by Aldous, a motivation for considering the above line-breaking construction is that it can be seen as a toy model of external diffusion limited aggregation (DLA). Recall that in the standard DLA model, say on \mathbb{Z}^2 , a subset \mathcal{A}_n is grown by recursively adding at each time a site on the boundary of \mathcal{A}_n according to the harmonic measure from infinity. It still remains a challenging open problem to understand the growth of \mathcal{A}_n , see [3, 10]. In our model the particles are now branches of varying size (we do not rescale the aggregate) and harmonic measure seen from infinity is replaced by uniform measure on the structure at time n . Our Theorem 1 can thus be interpreted as the fact that in this case the DLA aggregate does not grow arms towards infinity, and identifies its fractal dimension.

We finish this introduction by giving some elements of the proofs. In that aim, introduce the quantity

$$\mathbf{H}(\mathbf{a}) \quad := \quad \sum_{i=1}^{\infty} \frac{a_i^2}{A_i}.$$

When the sequence \mathbf{a} is bounded, we will see (Theorem 4) that condition $\mathbf{H}(\mathbf{a}) < \infty$ is equivalent to the convergence of the normalized length measure μ_n on \mathcal{T}_n towards a limiting random probability μ on \mathcal{T} . For connoisseurs, the latter is equivalent to the convergence of (\mathcal{T}_n, μ_n) to (\mathcal{T}, μ) in the Gromov-Prokhorov sense. In particular, condition $\mathbf{H}(\mathbf{a}) < \infty$ ensures that the height of a “typical” point of \mathcal{T} (i.e. sampled according to μ) is bounded. However it does not prevent \mathcal{T} from having very thin tentacles making it unbounded.

Under the hypotheses of Theorem 1, this phenomenon cannot happen thanks to an approximate scale invariance of the process. Roughly speaking, we prove that when $a_i \leq i^{-\alpha+o(1)}$, the subtree descending

from the i th branch is a random tree built by an aggregation process which is similar to the construction of the original tree except that it is scaled by a factor smaller than $i^{-\alpha+o(1)}$. This gives the first hint that the fractal dimension of \mathcal{T} is smaller than α^{-1} . On the other hand, when $A_i \geq i^{1-\alpha+o(1)}$ and $H(\mathbf{a}) < \infty$, the lower bound on the dimension is obtained using Frostman's theory by constructing a (random) measure nicely spread on \mathcal{T} . This role will be played by the limiting measure μ . To estimate the μ -measure of typical balls of radius $r > 0$ in \mathcal{T} (Lemma 13) we will compute the distribution of the distance of two typical points picked independently at random according to μ in \mathcal{T} , a.k.a. the two-point function (Lemma 14).

Under the hypotheses of Theorem 2, the upper bound of the dimension of the set of leaves is even true in a deterministic setting (Proposition 15), as well as the compactness, and is obtained by exhibiting appropriate coverings. The lower bound of the dimension is again obtained via Frostman's theory. A difficulty in this case is that the random measure μ is equal to the normalized length measure on \mathcal{T} (recall that the total length of \mathcal{T} is finite in this case). Hence, μ is supported by the skeleton of the tree, and not by the leaves. This forces us to introduce another random measure supported by the leaves of \mathcal{T} which captures its fractal behavior. This is done in the last section which is maybe the most technical part of this work.

Acknowledgments: The authors thank the organizers and the participants of the IXth workshop “Probability, Combinatorics and Geometry” at Bellairs institute (2014) where this work started. In particular, we are grateful to Omer Angel and Simon Griffiths for interesting discussions. We thank Frédéric Paulin for a question raised in 2008 which eventually yield to this work.

In this paper, unless mentioned, we only consider bounded sequences $(a_i)_{i \geq 1}$.

1 Tracking a uniform point

The goal of this section is to give a necessary and sufficient condition for the height of a typical point of \mathcal{T}_n (i.e. sampled according to the normalized length measure μ_n) to converge in distribution towards a finite random variable. For bounded sequence $(a_i)_{i \geq 1}$ this condition is just

$$H(\mathbf{a}) = \sum_{i=1}^{+\infty} \frac{a_i^2}{A_i} < \infty.$$

We will more precisely show that the above display is a necessary and sufficient condition for the convergence of the random measure μ_n towards a random probability measure μ carried by the limiting tree \mathcal{T} . We begin by introducing a piece of notation.

1.1 Notation

\mathbb{R} -trees as subsets of $\ell^1(\mathbb{R})$. We briefly recall here some definitions about \mathbb{R} -trees and refer to [4, 8] for precisions. An \mathbb{R} -tree is a metric space (\mathcal{T}, δ) such that for every $x, y \in \mathcal{T}$, there is a unique arc from x to y and this arc is isometric to a segment in \mathbb{R} . If $a, b \in \mathcal{T}$ we denote by $[[a, b]]$ the geodesic line segment between a and b in \mathcal{T} . The degree (or multiplicity) of a point $x \in \mathcal{T}$ is the number of connected components of $\mathcal{T} \setminus \{x\}$. A point of degree 1 is called a leaf and a point of degree larger than 3 is called a branch point.

Let $\mathbf{a} = (a_i)_{i \geq 1}$ be a sequence of positive reals, and $A_i = a_1 + \dots + a_i$, for $i \geq 1$, the associated sequence of partial sums. From \mathbf{a} , we build a sequence of random trees $(\mathcal{T}_n)_{n \geq 1}$ by grafting randomly closed segments (also called branches) of lengths $a_i, i \geq 1$ inductively as described in the introduction. To be more precise, we follow the initial approach of Aldous [1] and build \mathcal{T}_n as a subset of $\ell^1(\mathbb{R})$.

The tree \mathcal{T}_1 is $\{(x, 0, 0, \dots) : x \in [0, a_1]\}$ and recursively for every $n \geq 1$, conditionally on \mathcal{T}_n , we pick $(u_1^{(n)}, \dots, u_n^{(n)}, 0, 0, \dots) \in \mathcal{T}_n$ a uniform point on \mathcal{T}_n and set

$$\mathcal{T}_{n+1} := \mathcal{T}_n \cup \{(u_1^{(n)}, \dots, u_n^{(n)}, x, 0, 0, \dots) \in \ell^1(\mathbb{R}) : x \in [0, a_{n+1}]\}.$$

The point $\rho = (0, 0, \dots)$ will be seen as the root of the trees \mathcal{T}_n . With this point of view, the trees \mathcal{T}_n are increasing closed subsets of $\ell^1(\mathbb{R})$ and we can define their increasing union

$$\mathcal{T}^* = \bigcup_{n \geq 1} \mathcal{T}_n.$$

Note that $\mathcal{T}^* \subset \ell^1(\mathbb{R})$ will not be closed in general (or equivalently complete). We let \mathcal{T} denote its closure (or completion), which is therefore a random closed subset of $\ell^1(\mathbb{R})$. For us, \mathcal{T} and \mathcal{T}_n once endowed with their length metric δ , will be viewed as random \mathbb{R} -trees (recall that, in general, the completion of an \mathbb{R} -tree is an \mathbb{R} -tree – see e.g. [7]). In the rest of this article, we will be loose on the fact that $\mathcal{T}_n, \mathcal{T}$ are subsets of $\ell^1(\mathbb{R})$ and will use it only when necessary for technical proofs.

General notation. Let $(\mathcal{F}_n)_{n \geq 1}$ denote the associated filtration generated by $(\mathcal{T}_n)_{n \geq 1}$, and write \mathbf{b}_i for the segment or branch of index i which is seen as a subset of \mathcal{T}_n for each $n \geq i$. A moment of thought shows that $\mathcal{T} \setminus \mathcal{T}^*$ is only made of leaves of \mathcal{T} . We should stress that, although our main goal is to study some geometric properties of the sole tree \mathcal{T} , we will often need to work with its subtrees $\mathcal{T}_n, n \geq 1$. In that aim, we label the leaves of \mathcal{T}^* by order of apparition in the aggregation procedure, so that when observing \mathcal{T} , we also know \mathcal{T}_n , which is simply the subtree of \mathcal{T} spanned by the root and the leaves labeled $1, \dots, n, \forall n \geq 1$. This property is automatic when \mathcal{T}_n is constructed as a subset of $\ell^1(\mathbb{R})$ as before since the i th branch ranges over the i th coordinate of $\ell^1(\mathbb{R})$.

Besides, as already mentioned, we denote by μ_n the length measure on \mathcal{T}_n normalized by A_n^{-1} to make it a probability measure. Also, to lighten notation, we write $\text{ht}(x) = \delta(x, \rho)$ for the height of $x \in \mathcal{T}$.

Thanks to the nested structure of the trees $(\mathcal{T}_n)_{n \geq 1}$, for $k \geq 1$ and for any point $x \in \mathcal{T}$, we can make sense of $[x]_k$ the projection of x onto \mathcal{T}_k , that is the (unique) point of \mathcal{T}_k that minimizes the distance to x . If $A \subset \mathcal{T}$, for all $n \geq i$ we denote by

$$\mathcal{T}_n^{(i)}(A) = \{x \in \mathcal{T}_n : [x]_i \in A\}, \tag{1}$$

the subtree “descending from” A in \mathcal{T}_n . Similarly we let $\mathcal{T}^{(i)}(A) = \{x \in \mathcal{T} : [x]_i \in A\}$, the subtree “descending from” A in \mathcal{T} . Note that these definitions depend in general on the integer i . E.g.,

$$\mathcal{T}^{(2)}(\mathcal{T}_1) \subsetneq \mathcal{T}^{(1)}(\mathcal{T}_1) = \mathcal{T}.$$

Stems. A stem of a tree is a maximal open segment that contains no branch point. We will use a genealogical labeling of the stems of the trees $(\mathcal{T}_n)_{n \geq 1}$ by the ternary tree

$$\mathcal{G} = \bigcup_{i \geq 0} \{0, 1, 2\}^i,$$

with the usual genealogical order \preceq . Formally the first branch \mathbf{b}_1 is labeled by \emptyset . Once we graft a branch on it, it is split into three stems denoted (arbitrary) by $0, 1, 2$. Recursively, when the stem labeled $u \in \mathcal{G}$ is split into three by grafting a new branch on it, we denote $u0, u1, u2$ the three stems created. Here and later we implicitly identify a stem with its label. When \mathcal{T}_n is built after n graftings we denote by $\mathcal{G}_n \subset \mathcal{G}$ the set of all stems of \mathcal{T}_n .

When $u \in \mathcal{G}_i$ is a stem of \mathcal{T}_i we lighten the notation introduced in (1) and set

$$\mathcal{T}_n(u) := \mathcal{T}_n^{(i)}(u) \quad \text{and} \quad \mathcal{T}(u) := \mathcal{T}^{(i)}(u).$$

It is easy to check that these definition do not depend on i when u happens to belong to several \mathcal{G}_i . The last remark is also valid if u is the closure of a stem. We use the notation $\mathbf{L}(u)$ for the length of the stem u and introduce for $u \in \mathcal{G}_n$

$$\mathbf{a}(u) = (a_i(u))_{i \geq 1} = (0)_{1 \leq i \leq n-1} \cup \{\mathbf{L}(u)\} \cup (a_i \mathbb{1}_{\{a_i \text{ is grafted on } \mathcal{T}_{i-1}(u)\}})_{i \geq n+1}$$

for the sequence of lengths of branches that are recursively grafted onto the stem u or its descendants, with the convention that the first branch is the stem u appearing at time n . Note that $\mathbf{a}(u)$ corresponds to the lengths of branches used to construct $\mathcal{T}(u)$. We will sometimes need to consider a notion of height in these subtrees. Let $\bar{u} = u \cup \{a_u\} \cup \{b_u\}$ be the closure of u in \mathcal{T} , where a_u designs the vertex closest to the root. Then we define the height of a vertex $x \in \mathcal{T}(u)$ as the distance $\delta(a_u, x)$ and the height of the tree $\mathcal{T}(u)$ as the supremum of the distances $\delta(a_u, x)$ when x runs over $\mathcal{T}(u)$.

Remark 1. *Almost surely the set of branch-points of \mathcal{T} is dense in \mathcal{T} . Indeed, since the sequence $(a_i)_{i \geq 1}$ is bounded, $A_i \leq ci$ for some constant $c < \infty$ and all i . In particular*

$$\sum_{i \geq 1} \frac{1}{A_i} = \infty$$

and the Borel–Cantelli lemma implies that infinitely many branches will be grafted on each stem, almost surely. If $a_i \rightarrow 0$ we even have that the set of leaves of \mathcal{T} is dense in \mathcal{T} a.s..

1.2 Height of a random point

We begin with a simple key observation. Let $n \geq 2$ and conditionally on \mathcal{T}_n pick a point Y_n uniformly distributed according to the measure μ_n . Two cases may happen:

- with probability $1 - a_n/A_n$: the point Y_n belongs to the tree \mathcal{T}_{n-1} , that is $[Y_n]_{n-1} = Y_n$, and conditionally on this event $[Y_n]_{n-1}$ is uniformly distributed over \mathcal{T}_{n-1} ,
- with probability a_n/A_n : the point Y_n is located on the last branch \mathbf{b}_n grafted on \mathcal{T}_{n-1} . Conditionally on this event, Y_n is uniformly distributed on this branch and its projection $[Y_n]_{n-1}$ on the tree \mathcal{T}_{n-1} is independent of its location on the n th branch and is uniformly distributed on \mathcal{T}_{n-1} , given \mathcal{T}_{n-1} .

From this observation we deduce that $(\mathcal{T}_{n-1}, [Y_n]_{n-1}) = (\mathcal{T}_{n-1}, Y_{n-1})$ in distribution and more generally, $(\mathcal{T}_k, [Y_n]_k) = (\mathcal{T}_k, Y_k)$ in distribution for all $1 \leq k \leq n$. Note however an important subtlety: given the tree \mathcal{T}_n , the point $[Y_n]_{n-1}$ is *not* uniformly distributed on its subtree \mathcal{T}_{n-1} since $[Y_n]_{n-1}$ is located on a branch point of \mathcal{T}_n with probability a_n/A_n .

Reversing the process, it is possible to build a sequence $(\mathcal{T}_n, X_n)_{n \geq 1}$ recursively such that $[X_n]_k = X_k$ for all $k \leq n$ and such that $(\mathcal{T}_n, X_n) = (\mathcal{T}_n, Y_n)$ in law for every n . To do so, consider an independent sample $(U_i, V_i, i \geq 1)$ of i.i.d. uniform random variables on $(0, 1)$. Let first \mathcal{T}_1 be a segment of length a_1 , rooted at one end, and let X_1 be the point on this segment at distance $a_1 V_1$ from the root. We then proceed recursively and assume that the pair (\mathcal{T}_n, X_n) has been constructed. Then:

- if $U_{n+1} \leq a_{n+1}/A_{n+1}$, we branch a segment of length a_{n+1} on X_n to get \mathcal{T}_{n+1} and let X_{n+1} be the point on this segment at distance $a_{n+1} V_{n+1}$ from the branchpoint X_n ,
- if $U_{n+1} > a_{n+1}/A_{n+1}$, we branch a segment of length a_{n+1} at a point chosen uniformly (and independently of X_n) at random in \mathcal{T}_n , and set $X_{n+1} = X_n$.

Clearly, $[X_n]_k = X_k$ for $1 \leq k \leq n$ and it is easy to see by induction that (\mathcal{T}_n, X_n) and (\mathcal{T}_n, Y_n) have the same distribution for all $n \geq 1$. It is important to notice that in this coupling, the distance between X_n and the root ρ is non-decreasing, and more precisely that for any $n \geq m \geq 0$,

$$\delta(X_n, \mathcal{T}_m) = \delta(X_n, X_m) = \sum_{i=m+1}^n a_i V_i \mathbb{1}_{\{U_i \leq \frac{a_i}{A_i}\}}, \quad (2)$$

where we have set $X_0 = \mathcal{T}_0 = \rho$. Recalling the definition of $H(\mathbf{a})$ we see that $\lim_{n \rightarrow \infty} \mathbb{E}[\text{ht}(X_n)] = H(\mathbf{a})/2$. Therefore, when $H(\mathbf{a}) < \infty$, the sequence $(\text{ht}(X_n))$ converges and moreover (X_n) is a Cauchy sequence, by (2), almost surely. So, in this case, (X_n) converges a.s. in \mathcal{T} , by completeness. The converse is also true:

Proposition 3 (Finiteness of a typical height). *For bounded sequences $(a_i)_{i \geq 1}$,*

$$(X_n) \text{ converges in } \mathcal{T} \text{ a.s.} \iff H(\mathbf{a}) < \infty.$$

Moreover, when $H(\mathbf{a}) < \infty$, if $X := \lim_{n \rightarrow \infty} X_n$, we have

$$\mathbb{E}[e^{\lambda \text{ht}(X)}] \leq e^{\lambda H(\mathbf{a})}, \quad \text{for all } \lambda \in [0, (\sup_{i \geq 1} a_i)^{-1}].$$

Proof. By (2), the convergence of (X_n) is equivalent to the convergence of the series $\sum_i a_i V_i \mathbb{1}_{\{U_i \leq a_i/A_i\}}$ and so the first point follows from the classical three series theorem. To establish the exponential bound, note that for all $n \geq 1$,

$$\begin{aligned} \mathbb{E}[e^{\lambda \text{ht}(X_n)}] &= \prod_{i=1}^n \left(\frac{A_i - a_i}{A_i} + \frac{a_i}{A_i} \mathbb{E}[e^{\lambda a_i V_i}] \right) \\ &= \prod_{i=1}^n \left(\frac{A_i - a_i}{A_i} + \frac{a_i}{A_i} \frac{1}{\lambda a_i} (e^{\lambda a_i} - 1) \right). \end{aligned}$$

Then, since $\lambda a_i \leq 1$, we can use the bound $e^x \leq 1 + x + x^2$ valid for all $x \in [0, 1]$, and also $\log(1 + x) \leq x$ for $x \geq 0$, to get

$$\begin{aligned} \prod_{i=1}^n \left(\frac{A_i - a_i}{A_i} + \frac{a_i}{A_i} \frac{1}{\lambda a_i} (e^{\lambda a_i} - 1) \right) &\leq \prod_{i=1}^n \left(\frac{A_i - a_i}{A_i} + \frac{a_i}{A_i} (1 + \lambda a_i) \right) \\ &= \exp \left(\sum_{i=1}^n \log \left(1 + \lambda \frac{a_i^2}{A_i} \right) \right) \\ &\leq \exp \left(\lambda \sum_{i=1}^n \frac{a_i^2}{A_i} \right). \end{aligned}$$

Letting $n \rightarrow \infty$ we get the desired bound. \square

Remark 2. By equation (2) we get that $\mathbb{P}(X_n = X_{n_0}, \forall n \geq n_0) = A_{n_0}/A_\infty$ and so, with probability one, the sequence (X_n) is eventually constant if and only if $\sum_i a_i$ is convergent.

Remark 3. In the case of unbounded sequences $(a_i)_{i \geq 1}$ (not considered in this paper) the three series theorem shows that (X_n) converges a.s. iff there exists some $\varepsilon > 0$ such that

$$\sum_{i \geq 1} \frac{a_i}{A_i} \mathbb{1}_{\{a_i \geq \varepsilon\}} < \infty \quad \text{and} \quad \sum_{i \geq 1} \frac{a_i^2}{A_i} \mathbb{1}_{\{a_i \leq \varepsilon\}} < \infty.$$

Examples:

1. If the sum $\sum_{i \geq 1} a_i$ is finite, or if $a_i \leq i^{-\varepsilon + o(1)}$ for some $\varepsilon > 0$, then $H(\mathbf{a})$ is finite (see Lemma 22 (ii)) and so the tree \mathcal{T}_n has a typical height which remains bounded as $n \rightarrow \infty$. Proposition 9 and Proposition 15 actually state that in these cases the maximal height of the tree \mathcal{T}_n remains bounded as $n \rightarrow \infty$.

2. If $a_i \sim (\ln i)^{-\lambda}$ for some $\lambda \leq 1$ then $H(\mathbf{a}) = \infty$ and so the typical height of \mathcal{T}_n blows up. On the other hand, if $a_i \sim (\ln i)^{-\lambda}$ for some $\lambda > 1$ then $H(\mathbf{a}) < \infty$ and the typical height of \mathcal{T}_n thus remains bounded. In this case, we do not know whether the maximal height of \mathcal{T}_n remains stochastically bounded as $n \rightarrow \infty$.
3. Consider the sequence

$$a_i = i^{-1/2} + \mathbb{1}_{\{i \in \mathbb{N}^3\}} \quad \forall i \geq 1.$$

Clearly, $A_i \sim 2\sqrt{i}$ and $H(\mathbf{a}) < \infty$. Although the typical height of \mathcal{T}_n remains bounded, the tree \mathcal{T} is not compact since it contains an infinite number of branches of length greater than 1. (In fact, this tree is even unbounded, see Subsection 1.4.)

1.3 Convergence of the length measure μ_n

By Proposition 3 when $H(\mathbf{a}) = \infty$ the height of a random point in \mathcal{T}_n sampled according to μ_n tends in probability to ∞ . It follows that the sequence of probability measures (μ_n) cannot converge weakly in this context. However we will see that it does converge as soon as $H(\mathbf{a}) < \infty$. With no loss of generality, we assume in the sequel that the tree \mathcal{T} is built jointly with the sequence (X_n) , as explained in the previous section.

Theorem 4 (Convergence of the length measures). *Suppose that $H(\mathbf{a}) < \infty$. Then almost surely, there exists a probability measure μ on \mathcal{T} such that*

$$\mu_n \rightarrow \mu \quad \text{weakly as } n \rightarrow \infty.$$

Furthermore, conditionally on μ , the point $X = \lim_{n \rightarrow \infty} X_n$ is distributed according to μ almost surely and there is the dichotomy:

- if $\sum_i a_i = \infty$ then μ is a.s. supported by the leaves of \mathcal{T} ,
- if $\sum_i a_i < \infty$ then μ is a.s. supported by the skeleton of \mathcal{T} and coincides with the normalized length measure of \mathcal{T} .

To get a precise meaning of this theorem, recall that the trees $\mathcal{T}_n, n \geq 1$ and \mathcal{T} were actually constructed as closed subsets of $\ell^1(\mathbb{R})$. Hence, the random probability measures μ_n are just random variables with values in the Polish space of probability measures on $\ell^1(\mathbb{R})$ endowed with the Lévy-Prokhorov distance (which induces the weak convergence topology). Recall that the Lévy-Prokhorov distance on the probability measures of a metric space (E, d) is given by

$$d_{\text{LP}}(\mu, \nu) = \inf \left\{ \varepsilon > 0 : \nu(A) \leq \mu(A^{(\varepsilon)}) + \varepsilon \text{ and } \mu(A) \leq \nu(A^{(\varepsilon)}) + \varepsilon, \text{ for all Borel } A \subset E \right\},$$

and where $A^{(\varepsilon)} = \{y \in E : d(y, A) \leq \varepsilon\}$ is the ε -enlargement of A . To prove the first point of the theorem we will show that (μ_n) is a Cauchy sequence. We point out that this is not a direct consequence of Proposition 3. Indeed, as noticed in the previous section, given the tree \mathcal{T} , the variable X_n is *not* distributed according to μ_n since it is equal to a branch point of \mathcal{T} with a strictly positive probability.

The proof of Theorem 4 occupies the rest of this subsection. We start by introducing a family of martingales which will play an important role.

Mass martingales. Let $C \subset \mathcal{T}_i$ be measurable for \mathcal{F}_i and recall the notation $\mathcal{T}_n^{(i)}(C)$ for $n \geq i$ and $\mathcal{T}^{(i)}(C)$ introduced in (1). Set then $M_n(C) = \mu_n(\mathcal{T}_n^{(i)}(C))$ to simplify notation. Since the branches are grafted uniformly on the structure at each step, we have conditionally on \mathcal{F}_n

$$\begin{cases} M_{n+1}(C) = (A_n \cdot M_n(C) + a_{n+1})/A_{n+1} & \text{with proba. } M_n(C), \\ M_{n+1}(C) = A_n \cdot M_n(C)/A_{n+1} & \text{with proba. } 1 - M_n(C). \end{cases}$$

It readily follows that $(M_n(C))_{n \geq i}$ is a martingale with respect to $(\mathcal{F}_n)_{n \geq i}$ and since it takes values in $[0, 1]$, it converges almost surely to its limit $M(C) \in [0, 1]$. This limit $M(C)$ is the natural candidate for the value of $\mu(\mathcal{T}^{(i)}(C))$ of the possible limit μ of (μ_n) .

Remark 4 (Generalized Polya urn). *These martingales are also known as “generalized Polya urns” in the theory of reinforced processes. In general, it is a subtle question to discuss whether $M(C)$ can have atoms in $\{0, 1\}$, see [9]. However, in our context, since the sequence $(a_i)_{i \geq 1}$ is bounded, it follows from Pemantle’s work [9] that $M(C) \in (0, 1)$ almost surely when C and C^c have positive length measures. Let us emphasize an important consequence for us. Consider $C \subset \mathcal{T}_i$ with positive length measure and \mathcal{F}_i -measurable and let J be an infinite subset of \mathbb{N} . Then,*

$$\sum_{j \in J, j \geq i} M_j(C) = \infty \quad \text{a.s.}$$

and the conditional version of the Borel–Cantelli lemma implies that almost surely an infinite number of branches $\mathbf{b}_j, j \in J$ belong to the subtree $\mathcal{T}^{(i)}(C)$.

Lemma 5. *Assume $H(\mathbf{a}) < \infty$. Then almost surely, for any $\varepsilon > 0$, there exists (a random) n_0 such that*

$$\mu_n(\mathcal{T}_{n_0}^{(\varepsilon)}) \geq 1 - \varepsilon \quad \text{for all } n \geq 1.$$

Proof. We use the construction of (\mathcal{T}_n, X_n) of Section 1.2 and consider the stopping time (with respect to the filtration (\mathcal{F}_n)) defined by

$$\theta = \inf \left\{ n \geq 1 : \mu_n(\mathcal{T}_{n_0}^{(\varepsilon)}) < 1 - \varepsilon \right\}.$$

Fix now $\varepsilon > 0$ and a (deterministic) integer n_0 and note that

$$\begin{aligned} \mathbb{P}(\theta < \infty, \delta(X_\theta, X_{n_0}) \leq \varepsilon) &= \sum_{n \geq 1} \mathbb{E}[\mathbb{P}(\theta = n, \delta(X_n, X_{n_0}) \leq \varepsilon | \mathcal{F}_n)] \\ &\leq \sum_{n \geq 1} \mathbb{E}[\mathbb{1}_{\{\theta = n\}}] (1 - \varepsilon) = (1 - \varepsilon) \mathbb{P}(\theta < \infty) \end{aligned}$$

where we have used that the distribution of X_n given \mathcal{F}_n is μ_n , as well as the definition of θ , to get the second inequality. This yields

$$\begin{aligned} \varepsilon \cdot \mathbb{P}(\theta < \infty) &\leq \mathbb{P}(\theta < \infty, \delta(X_\theta, X_{n_0}) > \varepsilon) \\ &\leq \mathbb{P}(\delta(X, X_{n_0}) > \varepsilon) \\ &\stackrel{(2)}{=} \frac{1}{2\varepsilon} \sum_{i=n_0+1}^{\infty} \frac{a_i^2}{A_i}. \end{aligned}$$

Since the right-hand side can be made arbitrarily small by letting $n_0 \rightarrow \infty$, we get that almost surely, for every $\varepsilon > 0$ (rational say), there exists (a random) $n_0 \geq 1$ such that $\mu_n(\mathcal{T}_{n_0}^{(\varepsilon)}) \geq 1 - \varepsilon$ for all $n \geq 1$. \square

Lemma 6. *Assume $H(\mathbf{a}) < \infty$. Then almost surely (μ_n) is a Cauchy sequence for the Lévy-Prokhorov distance.*

Proof. For any $0 \leq k \leq n$, let $[\mu_n]_k$ be the measure μ_n projected onto \mathcal{T}_k , that is the push forward of μ_n by $x \mapsto [x]_k$. The following assertions hold almost surely. Fix $\varepsilon > 0$, it follows from the last lemma that there exists (a random) n_0 such that for all $n \geq 1$

$$d_{\text{LP}}(\mu_n; [\mu_n]_{n_0}) \leq \varepsilon. \tag{3}$$

Indeed, if Y_n is sampled according to μ_n then we have $\delta(Y_n, [Y_n]_{n_0}) \leq \varepsilon$ with probability at least $1 - \varepsilon$. Since $[Y_n]_{n_0}$ is distributed as $[\mu_n]_{n_0}$ this readily implies the (3). We then decompose \mathcal{T}_{n_0} into a finite number of \mathcal{F}_{n_0} -measurable pieces C_1, \dots, C_K of diameter less than ε (note that K is random). For each of these pieces recall the definition of the martingale $M_n(C_j)$ for $n \geq n_0$. In particular with our notation we have $M_n(C_j) = [\mu_n]_{n_0}(C_j)$. Next, note that when

$$\sum_{i=1}^K |M_n(C_i) - M_m(C_i)| \leq \varepsilon,$$

we can couple $X \sim [\mu_n]_{n_0}$ and $X' \sim [\mu_m]_{n_0}$ so that X and X' belong to the same set C_i with probability at least $1 - \varepsilon$. This implies that $d_{\text{LP}}([\mu_n]_{n_0}; [\mu_m]_{n_0}) \leq \varepsilon$. Since the martingales $(M_n(C_i))$ converge as $n \rightarrow \infty$, the last display is eventually fulfilled for n, m large enough. As a result, for n, m large enough

$$d_{\text{LP}}([\mu_n]_{n_0}; [\mu_m]_{n_0}) \leq \varepsilon.$$

Combining the last display with (3) we get that for n, m large enough, $d_{\text{LP}}(\mu_n; \mu_m) \leq 3\varepsilon$. Hence (μ_n) is almost surely Cauchy for the Lévy-Prokhorov distance on $\ell^1(\mathbb{R})$. \square

Proof of Theorem 4. The existence of the almost sure limit μ of (μ_n) is ensured by the previous lemma.

DISTRIBUTION OF X . Recall from Section 1.2 that $X_n \sim \mu_n$, given μ_n . In particular, for any $n \geq m$, $X_m = [X_n]_m$ is distributed according to $[\mu_n]_m$, given μ_n . Letting $n \rightarrow \infty$ and using the continuity of the projection on \mathcal{T}_m for the Lévy-Prokhorov distance, we obtain that $X_m \sim [\mu]_m$, given μ . Now, let $m \rightarrow \infty$. On the one hand, according to the arguments developed in the proof of Lemma 6, $[\mu]_m \rightarrow \mu$ almost surely for the Lévy-Prokhorov metric. On the other hand, $X_m \rightarrow X$ almost surely. It follows that $X \sim \mu$ almost surely given μ .

SUPPORT OF μ . Since $X \sim \mu$ almost surely given μ , we only need to show that $\mathbb{P}(X \text{ is a leaf of } \mathcal{T}) = 1$ or 0 according to $\sum_i a_i = \infty$ or $\sum_i a_i < \infty$. By the construction of X_n and X , we have

$$\mathbb{P}(X \text{ is a leaf in } \mathcal{T}) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}(X_n \notin \mathcal{T}_m).$$

If $\sum_i a_i = \infty$, by Remark 2, the sequence (X_n) escapes from any finite tree \mathcal{T}_m almost surely and so $\mathbb{P}(X \text{ is a leaf in } \mathcal{T}) = 1$. Conversely if $\sum_i a_i < \infty$, then $X_n = X$ eventually so $\mathbb{P}(X \text{ is a leaf in } \mathcal{T}) = 0$. In this case, (μ_n) converges clearly towards the normalized length measure on \mathcal{T} . \square

1.4 Boundedness of the whole tree

By Proposition 3 if the tree \mathcal{T} is bounded we must have $H(\mathbf{a}) < \infty$. We refine this a little:

Proposition 7. *A necessary condition for the tree \mathcal{T} to be bounded is that $a_i \rightarrow 0$ as $i \rightarrow \infty$.*

Proof. To see this, assume that there is a real number $\varepsilon > 0$ and an infinite subset J of \mathbb{N} such that $a_i \geq \varepsilon$ for all $i \in J$ (recall that the a_i are however supposed to be bounded). For each $i \in J$, let \mathbf{b}_i^+ denote the half part of the branch \mathbf{b}_i composed by the points at distance larger than $\varepsilon/2$ from the vertex of \mathbf{b}_i which is the closest to the root of \mathcal{T} . Then, by an argument similar to that of Remark 4, we know that almost surely, for each \mathbf{b}_i^+ , $i \in J$, there is an infinite number of branches \mathbf{b}_j , $j \in J$ that belong to its descending subtree. Iterating the argument, we see that there is a path in \mathcal{T} containing an infinite number of disjoint segments of lengths all greater than or equal to $\varepsilon/2$. Hence \mathcal{T} is unbounded. \square

Using a variation of the above argument we even get

Proposition 8. *Almost surely, we have*

$$\mathcal{T} \text{ is compact} \iff \mathcal{T} \text{ is bounded.}$$

Proof. The implication \Rightarrow is deterministically true. Notice that the event $\{\mathcal{T} \text{ is not compact}\}$ is an event contained in the tail σ -algebra generated by the gluings and so has probability 0 or 1. We suppose thus that \mathcal{T} is almost surely non-compact and will prove that it is almost surely non-bounded. We need a little notation. Fix $n \geq m$, the set $\mathcal{T}_n \setminus \mathcal{T}_m$ is a forest (a finite family of trees) whose highest tree is denoted by $\tau(m, n)$ (we add its root to make it complete). It is easy to see that conditionally on \mathcal{F}_m , the tree $\tau(m, n)$ is grafted on a uniform point of \mathcal{T}_m . By monotonicity the limit

$$\xi = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \text{ht}(\tau(m, n)) \in [0, \infty]$$

exists and is independent of \mathcal{F}_m for any $m \geq 0$. By the zero-one law ξ is thus deterministic. If $\xi = \infty$, the proof is finished. If not, we must anyway have $\xi > 0$, otherwise \mathcal{T} would be pre-compact hence compact by completeness. If by contradiction \mathcal{T} is bounded, then there exists k such that

$$\mathbb{P}(\text{ht}(\mathcal{T}_k) \geq \text{ht}(\mathcal{T}) - \xi/4) \geq 1/2. \quad (4)$$

We denote by C_k the \mathcal{F}_k -measurable part

$$C_k = \{x \in \mathcal{T}_k : \delta(\rho, x) \geq \text{ht}(\mathcal{T}_k) - \xi/4\}.$$

Then for any $m \geq 1$, consider the stopping time $\theta(m) = \inf\{n \geq m : \text{ht}(\tau(m, n)) > \xi/2\}$ which is almost surely finite by definition of ξ . We put $\theta^0 = k$ and θ^r the r -fold composition $\theta \circ \dots \circ \theta(k)$ to simplify notation. Recalling that for any $i \geq 0$, conditionally on \mathcal{F}_{θ^i} , the tree $\tau(\theta^i, \theta^{i+1})$ is grafted on a uniform point of \mathcal{T}_{θ^i} we get

$$\mathbb{P}\left(\bigcap_{i=0}^{\infty} \left\{ \tau(\theta^i, \theta^{i+1}) \text{ is not grafted on } \mathcal{T}_{\theta^i}^{(k)}(C_k) \right\}\right) = \mathbb{E}\left[\prod_{i=0}^{\infty} \left(1 - \mu_{\theta^i}(\mathcal{T}_{\theta^i}^{(k)}(C_k))\right)\right]. \quad (5)$$

The Remark 4 shows that $\mu_n(\mathcal{T}_n^{(k)}(C_k))$ is a.s. bounded away from 0 uniformly in n and so the last display is equal to 0. This leads to a contradiction with (4) since grafting $\tau(\theta^i, \theta^{i+1})$ onto $\mathcal{T}_{\theta^i}^{(k)}(C_k)$ increases the height of \mathcal{T}_k by at least $\xi/4$ (strictly). \square

We will see in the forthcoming Proposition 9 and Proposition 15 that sufficient conditions for the compactness of \mathcal{T} are either that $a_i \leq i^{-\alpha + o(1)}$ for some $\alpha \in (0, 1]$ or that the series $\sum_i a_i$ is convergent. But we do not have a necessary and sufficient condition for boundedness or equivalently compactness of the tree, hence the following question :

Open question 1. *Find a necessary and sufficient condition for \mathcal{T} to be bounded.*

As mentioned in the Introduction, this problem was solved by Amini et al. [2] for decreasing sequences \mathbf{a} : in these cases, with probability one, the tree \mathcal{T} is bounded if and only if $\sum_{i \geq 1} i^{-1} a_i < \infty$. Note that in general this condition cannot be sufficient for boundedness: in the Example 3 of Section 1.2 the sum $\sum_{i \geq 1} i^{-1} a_i$ is finite, but the corresponding tree is unbounded since a_i does not converge to 0.

2 Infinite length case

The goal of this section is to prove Theorem 1. We will first prove (under more general conditions than those of Theorem 1) that \mathcal{T} is compact using a covering argument which will also give the upper bound $\dim_{\text{H}}(\mathcal{T}) \leq 1/\alpha$. The lower bound on the Hausdorff dimension then follows from a careful study of the random measure μ introduced in Theorem 4 and, again, is valid under more general conditions than those of Theorem 1.

2.1 Compactness and upper bound

The main result of this subsection is the following:

Proposition 9. *Assume that $a_i \leq i^{-\alpha+o(1)}$ for some $\alpha \in (0, 1]$. Then, almost surely, the random tree \mathcal{T} is compact and its Hausdorff dimension is smaller than α^{-1} .*

We point out that we more generally know that the tree \mathcal{T} is compact, with a set of leaves of Hausdorff dimension less than α^{-1} , as soon as $a_i \leq i^{-\alpha+o(1)}$ for some $\alpha > 0$. This follows from the previous result, together with the forthcoming Proposition 15. That being said, we focus in the rest of this subsection on the proof of Proposition 9 and assume that $a_i \leq i^{-\alpha+o(1)}$ for $\alpha \in (0, 1]$. We note with Lemma 22 (ii) that this implies that

$$\sum_{i=n}^{\infty} \frac{a_i^2}{A_i} \leq n^{-\alpha+o(1)},$$

which will be repeatedly used in the sequel.

2.1.1 Rough scale invariance

We begin with a proposition which is a rough version of scale invariance. In words it says that the typical height of every subtree grafted on \mathcal{T}_n is smaller than $n^{-\alpha+o(1)}$. Combined with Proposition 3, it is the core of the proof of Proposition 9. For a stem u , recall the notation $\mathbf{a}(u)$ from Section 1.1.

Proposition 10. *If $a_i \leq i^{-\alpha+o(1)}$ for some $\alpha \in (0, 1]$, then, almost surely,*

$$\sup_{u \in \mathcal{G}_n} \mathbf{H}(\mathbf{a}(u)) \leq n^{-\alpha+o(1)}.$$

Proof. We first prove that the longest length of a stem of \mathcal{T}_n is smaller than $n^{-\alpha+o(1)}$. To see this, suppose by contradiction that a stem of length $n^{-\alpha+\varepsilon}$ is present in \mathcal{T}_n for some $\varepsilon > 0$. Provided that n is large enough, since $a_i \leq i^{-\alpha+o(1)}$, this stem must be part of a branch b_i (of length a_i) grafted at some time $i \leq n/2$. It thus means that we can find a part of length $n^{-\alpha+\varepsilon}/2$ of the branch b_i whose endpoints are exactly at distance $kn^{-\alpha+\varepsilon}/2$ and $(k+1)n^{-\alpha+\varepsilon}/2$ for some $k \geq 0$ from the extremity of b_i closest to root of \mathcal{T}_i which has not been hit by the grafting process between times $\lfloor n/2 \rfloor + 1$ and n . For each k , such an event has probability at most

$$\left(1 - \frac{n^{-\alpha+\varepsilon}}{2A_n}\right)^{n/2} \leq \exp(-n^{\varepsilon+o(1)}),$$

since $A_n \leq n^{1-\alpha+o(1)}$ because $a_i \leq i^{-\alpha+o(1)}$ and $\alpha \in (0, 1]$. Summing over all possibilities to choose such a part on some b_i for some $i \leq n$, we find that asymptotically the probability that there is a stem of length at least $n^{-\alpha+\varepsilon}$ in \mathcal{T}_n is bounded above by

$$\sum_{i \leq n/2} \left(\frac{2a_i}{n^{-\alpha+\varepsilon}} + 1 \right) \exp(-n^{\varepsilon+o(1)}) = \exp(-n^{\varepsilon+o(1)}).$$

We easily conclude by an application of Borel–Cantelli that

$$\sup_{u \in \mathcal{G}_n} \mathbf{L}(u) \leq n^{-\alpha+o(1)}. \tag{6}$$

To deduce from this the proposition, we need the following lemma.

Lemma 11. *Pick a stem u of \mathcal{T}_n , then, conditionally on \mathcal{T}_n , for any $\lambda \geq 0$ such that $\lambda L(u) < 1$ and $\lambda a_i < 1$ for all $i \geq n$, we have*

$$\mathbb{E}\left[e^{\lambda H(\mathbf{a}(u))} \mid \mathcal{F}_n\right] \leq \exp\left(2\lambda\left(L(u) + \sum_{i=n+1}^{\infty} \frac{a_i^2}{A_i}\right)\right).$$

Proof. For $i \geq 1$, let $A_i(u) = a_1(u) + \dots + a_i(u)$ and for $p \geq 1$, let

$$\Sigma_p = \sum_{i=1}^p \frac{a_i(u)^2}{A_i(u)}, \quad \text{so that } \Sigma_{\infty} = H(\mathbf{a}(u)),$$

with the convention that $\frac{a_i(u)^2}{A_i(u)} = 0$ if $a_i(u) = 0$. Next, let $\lambda \geq 0$ satisfy the assumptions of the statement. For $p \geq n$, since the branch a_{p+1} is grafted on $\mathcal{T}_p(u)$ with probability $A_p(u)/A_p$, we have

$$\begin{aligned} \mathbb{E}\left[e^{\lambda \Sigma_{p+1}} \mid \mathcal{F}_p\right] &= e^{\lambda \Sigma_p} \left(\frac{A_p - A_p(u)}{A_p} + \frac{A_p(u)}{A_p} e^{\lambda \frac{a_{p+1}^2}{A_p(u) + a_{p+1}}} \right) \\ &= e^{\lambda \Sigma_p} \left(1 + \frac{A_p(u)}{A_p} \left(e^{\lambda \frac{a_{p+1}^2}{A_p(u) + a_{p+1}}} - 1 \right) \right) \\ &\leq e^{\lambda \Sigma_p} \left(1 + 2\lambda \frac{a_{p+1}^2}{A_p(u) + a_{p+1}} \times \frac{A_p(u)}{A_p} \right). \end{aligned}$$

To go from the second to the third line, we have used that $\lambda \frac{a_{p+1}^2}{A_p(u) + a_{p+1}} \leq \lambda a_{p+1} \leq 1$ and that $e^x - 1 \leq 2x$ for $x \in [0, 1]$. Besides, since for a fixed $c > 0$, the function $x \mapsto x/(x+c)$ is increasing on $(0, \infty)$ and $A_p(u) \leq A_p$ we have that $\frac{A_p(u)}{(A_p(u) + a_{p+1})A_p} \leq \frac{1}{A_{p+1}}$, which finally leads to

$$\mathbb{E}\left[e^{\lambda \Sigma_{p+1}} \mid \mathcal{F}_p\right] \leq e^{\lambda \Sigma_p} \left(1 + 2\lambda \frac{a_{p+1}^2}{A_{p+1}} \right).$$

Note that we also have $\mathbb{E}[e^{\lambda \Sigma_n}] = e^{\lambda L(u)} \leq 1 + 2\lambda L(u)$. So, conditioning in cascades over all integers $p \geq n$, we obtain

$$\begin{aligned} \mathbb{E}[e^{\lambda H(\mathbf{a}(u))} \mid \mathcal{F}_n] &= \mathbb{E}[e^{\lambda \Sigma_{\infty}} \mid \mathcal{F}_n] \leq (1 + 2\lambda L(u)) \prod_{i=n+1}^{\infty} \left(1 + 2\lambda \frac{a_i^2}{A_i} \right) \\ &\leq \exp\left(2\lambda\left(L(u) + \sum_{i=n+1}^{\infty} \frac{a_i^2}{A_i}\right)\right). \end{aligned}$$

□

Coming back to the proof of Proposition 10, fix $\varepsilon > 0$ and consider n_{ε} such that $a_n \leq n^{-\alpha+\varepsilon}$ and $\sum_n^{\infty} \frac{a_i^2}{A_i} \leq n^{-\alpha+\varepsilon}$ for all $n \geq n_{\varepsilon}$ (n_{ε} exists by Lemma 22 (ii) and since $a_i \leq i^{-\alpha+o(1)}$). Then, for $m \geq n_{\varepsilon}$, let \mathcal{E}_m denote the event

$$\sup_{n \in \mathcal{G}_n} L(u) \leq n^{-\alpha+\varepsilon} \quad \text{for all } n \geq m.$$

By the first part of the proof, $\mathbb{P}(\mathcal{E}_m)$ converges to 1 as $m \rightarrow \infty$. Next, for a fixed $m \geq n_{\varepsilon}$ and all $n \geq m$, using a standard Markov exponential inequality and Lemma 11 with $\lambda = n^{\alpha-\varepsilon}$ on the event \mathcal{E}_m , we get

$$\mathbb{P}\left(H(\mathbf{a}(u)) \geq n^{-\alpha+2\varepsilon} \mid \mathcal{E}_m\right) \leq \frac{e^{-\lambda n^{-\alpha+2\varepsilon}} \mathbb{E}\left[\mathbb{E}\left[e^{\lambda H(\mathbf{a}(u))} \mathbb{1}_{\mathcal{E}_m} \mid \mathcal{F}_n\right]\right]}{\mathbb{P}(\mathcal{E}_m)} \leq \frac{e^{-\lambda n^{-\alpha+2\varepsilon} + 4\lambda n^{-\alpha+\varepsilon}}}{\mathbb{P}(\mathcal{E}_m)} \leq e^{-n^{\varepsilon+o(1)}}.$$

Since there are $2n-1$ stems in \mathcal{T}_n , the Borel–Cantelli lemma shows that conditionally on \mathcal{E}_m we have $\sup_{u \in \mathcal{G}_n} H(\mathbf{a}(u)) \leq n^{-\alpha+o(1)}$ almost surely. The conclusion follows, since $\mathbb{P}(\mathcal{E}_m) \rightarrow 1$ as $m \rightarrow \infty$. □

Remark 5. When $a_i \leq i^{-\alpha+\circ(1)}$ for some $\alpha > 1$ the statement of this proposition is no longer true. Indeed, in this case the length of the largest stem of \mathcal{T}_n is roughly of order $n^{-1} \gg n^{-\alpha}$.

2.1.2 Proof of Proposition 9

COMPACTNESS. Recall that \mathcal{T}_n and \mathcal{T} have been built as closed subsets of $\ell^1(\mathbb{R})$. Since the set of non-empty compact subspaces of $\ell^1(\mathbb{R})$ endowed with the Hausdorff distance (denoted here by δ_H) is complete, it suffices to show that

$$\sum_{i \geq 1} \delta_H(\mathcal{T}_{2^{i+1}}, \mathcal{T}_{2^i}) < \infty \quad \text{almost surely} \quad (7)$$

to get the almost sure compactness of \mathcal{T} . Note that $\delta_H(\mathcal{T}_{2^{i+1}}, \mathcal{T}_{2^i})$ is smaller than, or equal to, the maximal height of subtrees $\mathcal{T}_{2^{i+1}}(u)$ when u runs over \mathcal{G}_{2^i} (the subtrees $\mathcal{T}_n(u), \mathcal{T}(u)$ are defined in Section 1.1). To approximate the heights of these subtrees, we will throw 2^i independent uniform points in each of them and take the maximal height attained. Fix $\varepsilon > 0$ and let n_ε be such that $a_n \leq n^{\varepsilon-\alpha}$ for $n \geq n_\varepsilon$. For each $m \geq n_\varepsilon$, consider the event \mathcal{E}'_m on which

$$\sup_{u \in \mathcal{G}_n} H(\mathbf{a}(u)) \leq n^{\varepsilon-\alpha} \quad \text{for all } n \geq m.$$

By Proposition 10, $\mathbb{P}(\mathcal{E}'_m) \rightarrow 1$ as $m \rightarrow \infty$. It thus suffices to work conditionally on \mathcal{E}'_m .

So, fix $i \geq 1$ such that $2^i \geq m$, pick $u \in \mathcal{G}_{2^i}$ and let $H(u)$ denote the height of a random uniform point in $\mathcal{T}_{2^{i+1}}(u)$. By Proposition 3 with $\lambda = 2^{i(\alpha-\varepsilon)}$ we have

$$\begin{aligned} \mathbb{P}(H(u) \geq 2^{i(2\varepsilon-\alpha)} \mid \mathcal{E}'_m, \mathcal{F}_{2^i}) &\stackrel{\text{Markov}}{\leq} \frac{\mathbb{E} \left[e^{2^{i(\alpha-\varepsilon)} H(u)} \mathbb{1}_{\mathcal{E}'_m} \mid \mathcal{F}_{2^i} \right]}{\exp(2^{i(2\varepsilon-\alpha)} 2^{i(\alpha-\varepsilon)}) \mathbb{P}(\mathcal{E}'_m)} \\ &\leq \frac{\mathbb{E} \left[\mathbb{E} \left[e^{2^{i(\alpha-\varepsilon)} H(u)} \mathbb{1}_{\{H(\mathbf{a}(u)) \leq 2^{i(\varepsilon-\alpha)}\}} \mid \mathbf{a}(u), \mathcal{F}_{2^i} \right] \mid \mathcal{F}_{2^i} \right]}{\exp(2^{i\varepsilon}) \mathbb{P}(\mathcal{E}'_m)} \\ &\stackrel{\text{Prop. 3}}{\leq} \frac{\mathbb{E} \left[e^{2^{i(\alpha-\varepsilon)} H(\mathbf{a}(u))} \mathbb{1}_{\{H(\mathbf{a}(u)) \leq 2^{i(\varepsilon-\alpha)}\}} \mid \mathcal{F}_{2^i} \right]}{\exp(2^{i\varepsilon}) \mathbb{P}(\mathcal{E}'_m)} \leq \frac{e^1}{\exp(2^{i\varepsilon}) \mathbb{P}(\mathcal{E}'_m)}. \end{aligned} \quad (8)$$

To apply Proposition 3 in the third line we had to notice that conditionally on the sequence $\mathbf{a}(u)$, the tree $\mathcal{T}(u)$ is constructed from $\mathbf{a}(u)$ as \mathcal{T} is constructed from \mathbf{a} . In particular, according to the discussion preceding Proposition 3, the height of a uniform point in $\mathcal{T}_{2^{i+1}}(u)$ is stochastically smaller than the height of a uniform point in $\mathcal{T}(u)$, conditionally on $\mathbf{a}(u)$.

We now throw 2^i independent uniform points in each of the $2^{i+1} - 1$ subtrees $\mathcal{T}_{2^{i+1}}(u)$, for each $u \in \mathcal{G}_{2^i}$. Let \mathcal{B}_i denote the event “the maximal height attained by one of these $(2^{i+1} - 1) \cdot 2^i$ uniform points is larger than $2^{i(2\varepsilon-\alpha)}$ ”. By (8), conditionally on \mathcal{E}'_m , the probability of \mathcal{B}_i is bounded from above by

$$(2^{i+1} - 1) \cdot 2^i \frac{e^1}{\exp(2^{i\varepsilon}) \mathbb{P}(\mathcal{E}'_m)}.$$

The last quantity is summable in $i \geq 0$, hence by Borel–Cantelli we conclude that \mathcal{B}_i happens finitely many often, conditionally on \mathcal{E}'_m .

On the other hand, for each $u \in \mathcal{G}_{2^i}$, the total length of $\mathcal{T}_{2^{i+1}}(u)$ is smaller than $A_{2^{i+1}} \leq 2^{i(1-\alpha+\circ(1))}$. Hence when we throw independently 2^i uniform points in this subtree, the probability that none of these points is at distance less than $2^{i(2\varepsilon-\alpha)}$ of the maximal height is smaller than

$$\left(1 - \frac{2^{i(2\varepsilon-\alpha)}}{A_{2^{i+1}}}\right)^{2^i} \leq \exp\left(-2^i \frac{2^{i(2\varepsilon-\alpha)}}{2^{i(1-\alpha+\circ(1))}}\right) = \exp(-2^{i(2\varepsilon+\circ(1))}).$$

Even after multiplying the right-hand side by $2^{i+1} - 1$ the series is still summable, and so after another application of the Borel–Cantelli lemma, we can gather the last two results to deduce that almost surely (conditionally on \mathcal{E}'_m) for i large enough the heights of all subtrees $\mathcal{T}_{2^{i+1}}(u)$, $u \in \mathcal{G}_{2^i}$ is smaller than $2 \cdot 2^{i(2\varepsilon-\alpha)}$. Letting $m \rightarrow \infty$, this readily leads to (7).

UPPER BOUND ON THE HAUSDORFF DIMENSION. All the assertions in this paragraph hold almost surely. From the previous discussion, we deduce that conditionally on \mathcal{E}'_m the diameter of the trees $\mathcal{T}(u)$ for $u \in \mathcal{G}_{2^i}$ is smaller than $2^{i(3\varepsilon-\alpha)}$ for all i large enough. For those integers i , we thus obtain a covering of \mathcal{T} made of $2^{i+1} - 1$ balls of diameter $2^{i(4\varepsilon-\alpha)}$. This immediately implies that $\dim_{\text{H}}(\mathcal{T}) \leq 1/(\alpha - 4\varepsilon)$. Since $\varepsilon > 0$ was arbitrary and $\mathbb{P}(\mathcal{E}'_m) \rightarrow 1$, we indeed proved that $\dim_{\text{H}}(\mathcal{T}) \leq 1/\alpha$ a.s.

2.2 Lower bound via μ

Together with Proposition 9 and the fact $\dim_{\text{H}}(\mathcal{T}) \geq 1$, the following result implies Theorem 1.

Proposition 12. *Assume that $\text{H}(\mathbf{a}) < \infty$ and $A_n \geq n^{1-\alpha+\circ(1)}$ for $\alpha \in (0, 1)$. Then, the Hausdorff dimension of \mathcal{T} is larger than α^{-1} almost surely.*

Our approach relies on Frostman’s theory and the existence of the measure μ , the weak limit of the uniform measures μ_n which exists when $\text{H}(\mathbf{a}) < \infty$ by Theorem 4. Note that this result also applies to cases where we do not know if the tree \mathcal{T} is compact. E.g. the two hypotheses hold when $a_i = \ln(i)^{-\gamma}$ for some $\gamma > 1$, for all $\alpha \in (0, 1]$. In this case the Hausdorff dimension of the tree is therefore infinite a.s.

Remark 6. *When $\text{H}(\mathbf{a}) < \infty$ and $A_n \rightarrow \infty$, our proof below can easily be adapted to show that the Hausdorff dimension of $\text{Leaves}(\mathcal{T})$ is larger than 1 almost surely.*

The rest of this section is devoted to the proof of Proposition 12. So, consider a sequence $(a_i)_{i \geq 1}$ satisfying the two hypotheses of this proposition and recall that $\text{H}(\mathbf{a}) < \infty$ ensures the existence of the measure μ (Theorem 4). Then we know, by a result of Frostman [5, Theorem 4.13], that

$$\int_{\mathcal{T} \times \mathcal{T}} \frac{\mu(dx)\mu(dy)}{(\delta(x, y))^\gamma} < +\infty \Rightarrow \dim_{\text{H}}(\mathcal{T}) \geq \gamma$$

(we recall that δ denotes the distance on \mathcal{T}). Hence, given \mathcal{T} , consider two points picked uniformly and independently at random according to the measure μ , and let D denote their distance in \mathcal{T} . Clearly,

$$\mathbb{E}[D^{-\gamma}] = \mathbb{E}\left[\int_{\mathcal{T} \times \mathcal{T}} \frac{\mu(dx)\mu(dy)}{(\delta(x, y))^\gamma}\right],$$

from which we deduce that it is sufficient to prove that $\mathbb{E}[D^{-\gamma}] < \infty$ for all $\gamma \in (0, \alpha^{-1})$ to get the desired lower bound. This will be implied by the following lemma:

Lemma 13. *Under the conditions of Proposition 12, for all $\varepsilon > 0$, $\exists c_{\alpha, \varepsilon} > 0$ such that for all $r \in (0, 1]$,*

$$\mathbb{P}(D \leq r) \leq c_{\alpha, \varepsilon} r^{\frac{1}{\alpha} - \varepsilon}.$$

Consequently, $\mathbb{E}[D^{-\gamma}] < \infty$ for all $\gamma \in (0, \alpha^{-1})$.

To prove the last lemma we will compute exactly the (annealed) law of D in a similar fashion we computed the exact law of the height of a random point sampled according to μ . We then proceed to the proof of Lemma 13.

2.2.1 Description of the law of the two-point function

Lemma 14. Let $U_i, V_i, V'_i, i \geq 1$ be random variables independent and uniform on $[0, 1]$. The distribution of D is given by

$$\mathbb{E}[f(D)] = \sum_{k=1}^{\infty} \left[\left(\frac{a_k}{A_k} \right)^2 \prod_{j=k+1}^{\infty} \left(1 - \left(\frac{a_j}{A_j} \right)^2 \right) \right] \mathbb{E} \left[f \left(a_k |V_k - V'_k| + \sum_{i=k+1}^{\infty} a_i V_i \mathbb{1}_{\{U_i \leq 2 \frac{a_i}{A_i + a_i}\}} \right) \right]$$

for all measurable positive functions f .

Proof. Let $n \geq 2$ and conditionally on \mathcal{T} consider two points $Y_n^{(1)}$ and $Y_n^{(2)} \in \mathcal{T}_n$ independent and distributed according to μ_n . We let D_n denote their distance.

- With probability $(1 - \frac{a_n}{A_n})^2$ these two points belong to \mathcal{T}_{n-1} and conditionally on this event they are independent, uniform on \mathcal{T}_{n-1} . On this event we thus have $D_n \stackrel{(d)}{=} D_{n-1}$.
- With probability $2(1 - \frac{a_n}{A_n})(\frac{a_n}{A_n})$ only one of these points belongs to the n th branch. Conditionally on this event, the point in question is uniformly distributed on the last branch and the remaining point is independent and uniform on \mathcal{T}_{n-1} . Moreover the projection of these two points onto \mathcal{T}_{n-1} yields a pair of independent points uniformly distributed over \mathcal{T}_{n-1} . On this event we thus have $D_n \stackrel{(d)}{=} D_{n-1} + a_n V_n$ where in the right side, V_n is uniform on $(0, 1)$ and independent of D_{n-1} .
- Finally, with probability $(\frac{a_n}{A_n})^2$ these two points belong to the n th branch. Conditionally on this event they are uniform, independent on this branch, and thus we can write $D_n = a_n |V_n - V'_n|$ where V_n and V'_n are independent and both uniform on $(0, 1)$.

Noticing that for $n \geq 2$

$$\frac{2(1 - \frac{a_n}{A_n})(\frac{a_n}{A_n})}{1 - (\frac{a_n}{A_n})^2} = \frac{2a_n}{A_n + a_n},$$

it follows from the previous discussion that the law of D_n is described as follows:

$$\begin{aligned} & \text{for } k \in \{1, 2, \dots, n\} \text{ with probability } \left(\frac{a_k}{A_k} \right)^2 \prod_{i=k+1}^n \left(1 - \left(\frac{a_i}{A_i} \right)^2 \right) \\ & \text{we have } D_n = a_k |V_k - V'_k| + \sum_{i=k+1}^n a_i V_i \mathbb{1}_{\{U_i \leq 2 \frac{a_i}{A_i + a_i}\}}, \end{aligned}$$

where the variables $U_i, V_i, V'_i, 1 \leq i \leq n$ are all independent and uniform on $[0, 1]$ (we use the convention that the sum over the empty set is 0, whereas the product over the empty set is 1). From Theorem 4, we get that $D_n \rightarrow D$ in distribution so that passing to the limit, we get a similar description of the law of D . In this last step, it is crucial that the series $\sum_k (\frac{a_k}{A_k})^2$ converges to ensure that $\mathbb{P}(D = \infty) = 0$. We check in Lemma 22 that such a series is always convergent. \square

2.2.2 Proof of Lemma 13

Fix $\varepsilon \in (0, 1)$ and let $r \in (0, 1]$. By Lemma 14 we have

$$\begin{aligned} \mathbb{P}(D \leq r) &= \sum_{k=1}^{\infty} \left[\left(\frac{a_k}{A_k} \right)^2 \prod_{j=k+1}^{\infty} \left(1 - \left(\frac{a_j}{A_j} \right)^2 \right) \right] \mathbb{P} \left(a_k |V_k - V'_k| + \sum_{i=k+1}^{\infty} a_i V_i \mathbb{1}_{E_i} \leq r \right) \\ &\leq \sum_{k=\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor + 1}^{+\infty} \left(\frac{a_k}{A_k} \right)^2 \mathbb{P}(a_k |V_k - V'_k| \leq r) \\ &\quad + \sum_{k=1}^{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor} \left(\frac{a_k}{A_k} \right)^2 \mathbb{P}(a_k |V_k - V'_k| \leq r) \prod_{i=k+1}^{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor} \mathbb{P}(a_i V_i \mathbb{1}_{E_i} \leq r) \left(1 - \left(\frac{a_i}{A_i} \right)^2 \right), \end{aligned}$$

where we have set $E_i = \left\{ U_i \leq 2 \frac{a_i}{A_i + a_i} \right\}$ to improve the presentation. Then, note that

$$\mathbb{P}(a_i V_i \mathbb{1}_{E_i} \leq r) = 1 - \frac{2a_i}{A_i + a_i} + \frac{2a_i}{A_i + a_i} \times \frac{r}{a_i} \leq \frac{A_{i-1}^2}{A_i^2} \times \frac{A_i^2}{A_i^2 - a_i^2} \times \left(1 + \frac{2r}{A_{i-1}} \right),$$

which leads us to

$$\prod_{i=k+1}^{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor} \mathbb{P}(a_i V_i \mathbb{1}_{E_i} \leq r) \left(1 - \left(\frac{a_i}{A_i} \right)^2 \right) \leq \prod_{i=k+1}^{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor} \frac{A_{i-1}^2}{A_i^2} \times \prod_{i=k+1}^{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor} \left(1 + \frac{2r}{A_{i-1}} \right).$$

But the second product in the right-hand side is bounded from above by a constant independent of k and $r \in (0, 1]$. Indeed, using that $\ln(1+x) \leq x$ for positive x , we get that

$$\prod_{i=k+1}^{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor} \left(1 + \frac{2r}{A_{i-1}} \right) \leq \exp \left(2r \sum_{i=k+1}^{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor} \frac{1}{A_{i-1}} \right) \leq \exp \left(2r(r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}})^{\alpha + o(1)} \right),$$

where we have used the assumption on the lower bound of A_n for the second inequality (here the notation \circ refers to the convergence of r towards 0). Finally, we have proved the existence of a finite constant C independent of $r \in (0, 1]$ such that

$$\mathbb{P}(D \leq r) \leq \sum_{k=\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor + 1}^{+\infty} \left(\frac{a_k}{A_k} \right)^2 \times \frac{2r}{a_k} + C \sum_{k=1}^{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor} \left(\frac{a_k}{A_k} \right)^2 \times \frac{2r}{a_k} \times \frac{A_k^2}{A_{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor}^2}.$$

By Lemma 22 (iii), the first sum in the right-hand side is smaller than $r^{\frac{1}{\alpha} - \frac{(1-\alpha)\varepsilon}{2} + o(1)}$. So we finally get,

$$\begin{aligned} \mathbb{P}(D \leq r) &\leq r^{\frac{1}{\alpha} - \frac{(1-\alpha)\varepsilon}{2} + o(1)} + \frac{2rC}{A_{\lfloor r^{-\frac{1}{\alpha} + \frac{\varepsilon}{2}} \rfloor}} \\ &\leq r^{\frac{1}{\alpha} - \frac{(1-\alpha)\varepsilon}{2} + o(1)}. \end{aligned}$$

3 Finite length case

The goal of this section is to prove Theorem 2. As in the previous section, we will first prove the compactness and the upper bound of the Hausdorff dimension, which hold in a more general (and even deterministic) setting than that of Theorem 2. The lower bound on the dimension is more technical than in the previous section and requires the construction of a new measure supported by the leaves of \mathcal{T} .

3.1 Deterministic results in the finite length case

The following proposition does not depend on the fact that the new branches are grafted uniformly on the pre-existing tree, but just on the asymptotic behavior of the sequence $(a_i, i \geq 1)$. So, in this subsection, and only in this subsection, \mathcal{T} designs the completion of a tree built by grafting the branches b_i of lengths a_i iteratively, without any explicit rules on where the branches are glued. We denote by $\text{Leaves}(\mathcal{T})$ the set of leaves of \mathcal{T} .

Proposition 15. *If $\sum_{i=1}^{\infty} a_i < \infty$, the tree \mathcal{T} is compact and of Hausdorff dimension 1. Moreover,*

$$\dim_{\text{H}}(\text{Leaves}(\mathcal{T})) \leq \gamma \quad \text{as soon as} \quad \sum_{i=1}^{\infty} a_i^{\gamma} < \infty.$$

Proof. We start with the proof of the upper bound of the Hausdorff dimension of the leaves and assume that $\sum_{i \geq 1} a_i^\gamma < \infty$ for some $\gamma \leq 1$. Since the set of leaves of \mathcal{T}^* is at most countable, its Hausdorff dimension is 0. To get the expected upper bound, we thus only need to get an upper bound for the Hausdorff dimension of $\mathcal{T} \setminus \mathcal{T}^*$.

In that aim, fix $\varepsilon > 0$ and let n_ε be such that $\sum_{i > n_\varepsilon} a_i \leq \varepsilon$. Consider then the decomposition of $\mathcal{T} \setminus \mathcal{T}_{n_\varepsilon}$ into connected components and note that the set of closures of these components forms a (at most) countable set of closed subtrees of \mathcal{T} , that covers $\mathcal{T} \setminus \mathcal{T}^*$. The intersection of each of these subtrees with $\mathcal{T}_{n_\varepsilon}$ is reduced to a unique point, the root of the subtree (different subtrees may have the same root – recall that we have no explicit rule of gluing). We denote by \mathcal{R}_ε this set of roots, and, for all $r \in \mathcal{R}_\varepsilon$, by $\mathcal{T}_{n_\varepsilon}^{(r)}$ the union of subtrees descending from it, which is also a tree. We then let \mathcal{I}_r be the set of integers i such that the segment b_i belongs to the subtree $\mathcal{T}_{n_\varepsilon}^{(r)}$. Clearly, this subtree has a diameter smaller than $\sum_{i \in \mathcal{I}_r} a_i$ which is itself smaller than ε , by definition of n_ε .

The collection of subtrees $\mathcal{T}_{n_\varepsilon}^{(r)}$, $r \in \mathcal{R}_\varepsilon$ therefore forms an at most countable covering of $\mathcal{T} \setminus \mathcal{T}^*$ with sets of diameter less than ε . We have

$$\sum_{r \in \mathcal{R}_\varepsilon} \left(\sum_{i \in \mathcal{I}_r} a_i \right)^\gamma \leq \sum_{r \in \mathcal{R}_\varepsilon} \sum_{i \in \mathcal{I}_r} a_i^\gamma \leq \sum_{i \geq 1} a_i^\gamma < \infty,$$

where the first inequality holds since $\gamma \leq 1$ and the second since the sets \mathcal{I}_r , $r \in \mathcal{R}_\varepsilon$ are disjoint. Hence the γ -dimensional Hausdorff measure of $\mathcal{T} \setminus \mathcal{T}^*$ is finite and its Hausdorff dimension smaller than γ (almost surely).

We now turn to the compactness of \mathcal{T} under the sole assumption $\sum_{i \geq 1} a_i < \infty$. We consider $\varepsilon > 0$ and use the notation introduced above. The tree $\mathcal{T}_{n_\varepsilon}$ is clearly compact and we let $B(x_n, \varepsilon)$, $n \leq N_\varepsilon$ be a finite collection of open balls of radius ε that covers it. Besides, as noticed above, all $x \in \mathcal{T} \setminus \mathcal{T}_{n_\varepsilon}$ is at distance at most ε from an element of \mathcal{R}_ε . Consequently the collection of open balls $B(x_n, 2\varepsilon)$, $n \leq N_\varepsilon$ of radius 2ε covers \mathcal{T} . Hence \mathcal{T} is pre-compact and thus compact by completeness. \square

3.2 Lower bound for the Hausdorff dimension of the leaves

In this section we assume the existence of $\alpha > 1$ such that

$$(D_\alpha) \quad a_i \leq i^{-\alpha+o(1)} \quad \text{and} \quad a_i + a_{i+1} + \dots + a_{2i} = i^{1-\alpha+o(1)}.$$

In particular, by Proposition 15, the tree \mathcal{T} is compact and the Hausdorff dimension of its set of leaves is bounded above by $1/\alpha$ (almost surely). The following result is the complement to obtain the statement of Theorem 2.

Proposition 16. *Under (D_α) , almost surely,*

$$\dim_H(\text{Leaves}(\mathcal{T})) \geq 1/\alpha.$$

To get this lower bound, we will show that for any $\varepsilon > 0$ we can construct, with a probability at least $1 - \varepsilon$, a (random) probability measure π supported by the set of leaves of \mathcal{T} such that for every $x \in \mathcal{T}$

$$\limsup_{r \rightarrow 0} \frac{\pi(B(x, r))}{r^{\frac{1}{\alpha} - \varepsilon}} = 0, \tag{9}$$

where $B(x, r)$ denotes the open ball in \mathcal{T} of radius r centered at x . By standard results on Hausdorff dimensions (see e.g. [5, Proposition 4.9]), this will entail that $\dim_H(\text{Leaves}(\mathcal{T})) \geq \alpha^{-1} - \varepsilon$ with probability at least $1 - \varepsilon$. (Proposition 4.9 in [5] is stated for subsets of \mathbb{R}^n , but, clearly, its proof also holds for any metric space.) Since $\varepsilon > 0$ is arbitrary, this will prove Proposition 16.

From now on, $\varepsilon \in (0, 1/\alpha)$ is fixed. Rather than tempting to construct a “uniform” measure on the leaves of \mathcal{T} , the support of π will be a strict subset of $\text{Leaves}(\mathcal{T})$. To construct this measure, we need some more notation.

Subsets of good branches. For $i \geq 1$, we say that the branch \mathbf{b}_i , of length a_i , is “good” if $i^{-\alpha-\varepsilon} \leq a_i$. In other words, a good branch is not too small when it appears (it cannot be larger than $i^{-\alpha+\varepsilon}$ eventually according to (D_α)). For $n \geq 1$, let

$$G_n = \{i \in [n, 2n] : \mathbf{b}_i \text{ is good}\} \quad \text{and} \quad \ell_n = \sum_{i \in G_n} a_i,$$

ℓ_n being the total length of good branches of index between n and $2n$. It is easy to see that under assumption (D_α)

$$\#G_n = n^{1+o(1)} \quad \text{and} \quad \ell_n = n^{1-\alpha+o(1)}. \quad (10)$$

Let now $1 = n_1 < n_2 < n_3 \dots$ be integers such that $n_{k+1} > 2n_k$ for all $k \geq 1$. Later we will need to do some additional assumptions on the integers n_k ’s ensuring that they grow sufficiently fast, but for the moment we stay on this. For $\mathbf{b}_i, \mathbf{b}_j$ two good branches with indices $1 \leq j < i$, we write $\mathbf{b}_i \rightarrow \mathbf{b}_j$ if \mathbf{b}_i is directly grafted on \mathbf{b}_j . We let $\mathcal{B}_1 = \mathbf{b}_1$ and for $k \geq 2$ we define recursively the subsets \mathcal{B}_k of \mathcal{T} , by deciding that \mathcal{B}_k is made of the good branches \mathbf{b}_{i_k} , $n_k \leq i_k \leq 2n_k$ that are grafted on (good) branches of \mathcal{B}_{k-1} . This leads to branches of the form

$$\mathbf{b}_{i_k} \rightarrow \mathbf{b}_{i_{k-1}} \rightarrow \dots \rightarrow \mathbf{b}_{i_2} \rightarrow \mathbf{b}_1 \quad \text{with} \quad n_\ell \leq i_\ell \leq 2n_\ell \quad \text{for every } 2 \leq \ell \leq k.$$

Note that the sets $\mathcal{B}_k, k \geq 1$ may be empty. Slightly changing the notation introduced in Section 1.1, we let

$$\mathcal{T}(\mathbf{b}_i) = \{x \in \mathcal{T} : [x]_i \in \mathbf{b}_i\}$$

denote the subtree descending from \mathbf{b}_i and

$$\mathcal{T}(\mathcal{B}_k) = \bigcup_{i: \mathbf{b}_i \in \mathcal{B}_k} \mathcal{T}(\mathbf{b}_i).$$

Remark that $\mathcal{T}(\mathcal{B}_{k+1}) \subset \mathcal{T}(\mathcal{B}_k)$ for all $k \geq 1$. Conditionally on the event $\{\mathcal{B}_k \neq \emptyset, \forall k \geq 1\}$, let now π_k denote the normalized length measure on \mathcal{B}_k . We will see later, choosing the n_k ’s adequately, that the probability of this event can be made arbitrary close to 1 and that *the measure π will be obtained as a (subsequential) limit of $(\pi_k)_{k \geq 1}$* . Remark that conditionally on $\{\mathcal{B}_k \neq \emptyset, \forall k \geq 1\}$, the family $(\pi_k)_{k \geq 1}$ is a sequence of probability measures on a compact space, hence it admits at least one subsequential limit. We begin with a simple lemma.

Lemma 17. *Almost surely, conditionally on $\{\mathcal{B}_k \neq \emptyset, \forall k \geq 1\}$ (and provided that this event has a positive probability) any subsequential limit ϖ of $(\pi_k)_{k \geq 0}$ is supported by the set of leaves of \mathcal{T} .*

Proof. Clearly, $\delta(\mathcal{T}(\mathcal{B}_{k+1}), \mathcal{T}(\mathcal{B}_k)^c) > 0$ almost surely for all $k \geq 1$. Hence we can find an open set \mathcal{O}_k containing $\mathcal{T}(\mathcal{B}_k)^c$ such that $\pi_j(\mathcal{O}_k) = 0$ for all $j \geq k+1$, a.s. By the Portmanteau theorem, it follows that $\varpi(\mathcal{O}_k) = 0$ and so, almost surely,

$$\text{Supp}(\varpi) \subset \bigcap_{k \geq 1} \mathcal{T}(\mathcal{B}_k).$$

Since $\mathcal{T}(\mathcal{B}_k) \subset \mathcal{T} \setminus \mathcal{T}_{n_k-1}$ for all k , the right-hand side is a subset of $\mathcal{T} \setminus \mathcal{T}^*$. □

3.2.1 Lengths estimates.

Before embarking into the proof of Proposition 16, we have to set up some estimates on the total length of descendants in \mathcal{B}_{k+1} of a given subset of \mathcal{B}_k and also to check that the distance between most branches of \mathcal{B}_k is not too small provided that the sequence (n_k) grows sufficiently fast. This is the goal of this subsection. Once this will be done, we will see in the next subsection how to use this to show that when the sequence (n_k) grows sufficiently fast, the number of branches composing \mathcal{B}_k is roughly of order n_k whereas their lengths are of order $n_k^{-\alpha}$. This is a first hint that any subsequential limit of (π_k) should satisfy (9). Of course, we will need to control our approximations and the material to do that is developed here. We start with some estimates of the total length of good branches indexed by G_n that are grafted on a given subset of \mathcal{T}_{n-1} , $n \geq 1$.

Lemma 18. *Let $n \geq 2$ and consider a subset $S \subset \mathcal{T}_{n-1}$ measurable with respect to \mathcal{F}_{n-1} . Denote by \mathcal{X} the total length of the branches indexed by G_n that are (directly) grafted on S .*

(i) *Then for every $\eta \in (0, 1)$ we have*

$$\mathbb{P} \left(\left| \mathcal{X} - \frac{\ell_n |S|}{A_\infty} \right| \geq \eta \frac{\ell_n |S|}{A_\infty} \right) \leq \frac{n^{-c+\circ(1)}}{|S|\eta^2}, \quad \text{with } c = 1 \wedge (\alpha - 1) > 0.$$

(ii) *Fix $\delta > 0$ and $m \in \mathbb{N}$. Then, for all n large enough and then for all subsets S such that $|S| \geq n^{-1+\delta}$,*

$$\mathbb{E}[\mathcal{X}^m] \leq C_m (|S| \ell_n)^m,$$

where C_m depends only on m .

Proof. By construction, the random variable \mathcal{X} can be written as follows:

$$\mathcal{X} = \sum_{i \in G_n} a_i \mathbb{1}_{\left\{U_i \leq \frac{|S|}{A_{i-1}}\right\}},$$

where $(U_i)_{i \geq 1}$ is a sequence of independent random variables uniformly distributed on $(0, 1)$. In particular, $\mathbb{E}[\mathcal{X}] = \sum_{i \in G_n} \frac{a_i |S|}{A_{i-1}}$.

(i) Consider temporarily the variable $\tilde{\mathcal{X}} = \sum_{i \in G_n} a_i \mathbb{1}_{\left\{U_i \leq \frac{|S|}{A_\infty}\right\}}$ instead of \mathcal{X} . Clearly, $\mathbb{E}[\tilde{\mathcal{X}}] = \ell_n |S| / A_\infty$ and

$$\text{Var}(\tilde{\mathcal{X}}) = \sum_{i \in G_n} a_i^2 \text{Var} \left(\mathbb{1}_{\left\{U_i \leq \frac{|S|}{A_\infty}\right\}} \right) = \sum_{i \in G_n} a_i^2 \left(\frac{|S|}{A_\infty} \right) \left(1 - \frac{|S|}{A_\infty} \right) \stackrel{(D_\alpha)}{\leq} |S| n^{1-2\alpha+\circ(1)}.$$

On the other hand, $A_\infty - A_n = n^{1-\alpha+\circ(1)}$, again by (D_α) , and so

$$\mathbb{E} \left[|\mathcal{X} - \tilde{\mathcal{X}}| \right] = \sum_{i \in G_n} a_i \frac{|S|}{A_\infty} \frac{(A_\infty - A_{i-1})}{A_{i-1}} = n^{1-\alpha+\circ(1)} \ell_n |S|.$$

This leads to

$$\begin{aligned} \mathbb{P} \left(\left| \mathcal{X} - \frac{\ell_n |S|}{A_\infty} \right| \geq 2\eta \frac{\ell_n |S|}{A_\infty} \right) &\leq \mathbb{P} \left(\left| \tilde{\mathcal{X}} - \frac{\ell_n |S|}{A_\infty} \right| \geq \eta \frac{\ell_n |S|}{A_\infty} \right) + \mathbb{P} \left(|\mathcal{X} - \tilde{\mathcal{X}}| \geq \eta \frac{\ell_n |S|}{A_\infty} \right) \\ &\leq \frac{\text{Var}(\tilde{\mathcal{X}})}{\eta^2 \ell_n^2 |S|^2 / A_\infty^2} + \frac{\mathbb{E}[|\mathcal{X} - \tilde{\mathcal{X}}|]}{\eta \ell_n |S| / A_\infty} \\ &\leq \frac{n^{-1+\circ(1)}}{|S|\eta^2} + \frac{n^{1-\alpha+\circ(1)}}{\eta}. \end{aligned}$$

(ii) Next, let $i_1, \dots, i_{\#G_n}$ denote the indices of integers $i \in G_n$. We have for all integers $m \geq 1$,

$$\begin{aligned} \mathbb{E}[\mathcal{X}^m] &= \sum_{\substack{n_{i_1}, \dots, n_{i_{\#G_n}} : \\ n_{i_1} + \dots + n_{i_{\#G_n}} = m}} \binom{m}{n_{i_1}, \dots, n_{i_{\#G_n}}} \prod_{j=1}^{\#G_n} a_{i_j}^{n_{i_j}} \mathbb{E} \left[\left(\mathbb{1}_{\{U_{i_j} \leq \frac{|S|}{A_{i_j}-1}\}} \right)^{n_{i_j}} \right] \\ &\leq m! \sum_{\substack{n_{i_1}, \dots, n_{i_{\#G_n}} : \\ n_{i_1} + \dots + n_{i_{\#G_n}} = m}} \left(\frac{|S|}{A_\infty} \right)^{\#\{j: n_{i_j} \geq 1\}} \prod_{j=1}^{\#G_n} a_{i_j}^{n_{i_j}}, \end{aligned}$$

where we have simply bounded the multinomial term by $m!$. Observe that for every $\#G_n$ -tuple involved in the sum, by (D_α) ,

$$\prod_{j=1}^{\#G_n} a_{i_j}^{n_{i_j}} \leq n^{-m(\alpha + o(1))}.$$

Then, by grouping the $\#G_n$ -tuples according to the number of non-zero terms they contain, we get the existence of a constant c_m depending only on m such that

$$\mathbb{E}[\mathcal{X}^m] \leq m! \sum_{\substack{n_{i_1}, \dots, n_{i_{\#G_n}} \in \{0, 1\} : \\ n_{i_1} + \dots + n_{i_{\#G_n}} = m}} \left(\frac{|S|}{A_1} \right)^m \prod_{j=1}^{\#G_n} a_{i_j}^{n_{i_j}} + c_m \sum_{p=1}^{(m-1) \wedge \#G_n} \binom{\#G_n}{p} |S|^p n^{-m(\alpha + o(1))}.$$

Note that the first term in the right-hand side may be null (if $\#G_n < m$) and is anyway always smaller than $(A_1^{-1}|S|\ell_n)^m$. Now, noticing that $\binom{\#G_n}{p} \leq (\#G_n)^p$ and using that $|S| \geq n^{-1+\delta}$, we see by (10) that

$$\binom{\#G_n}{p} |S|^p n^{-m(\alpha + o(1))} \leq (|S|\ell_n)^m,$$

provided that n is large enough, independently of $p, |S|$. This is sufficient to conclude. \square

Corollary 19. *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ with $f(n) > 2n$ for all $n \geq 1$, such that if the sequence $(n_k)_{k \geq 1}$ satisfies $n_{k+1} \geq f(n_k)$ for all $k \geq 1$, then with probability at least $1 - \varepsilon$,*

$$|\mathcal{T}(\mathbf{b}_i) \cap \mathcal{B}_{k+1}| \in \left[(1 - 2^{-k}) \frac{a_i \ell_{n_{k+1}}}{A_\infty}, (1 + 2^{-k}) \frac{a_i \ell_{n_{k+1}}}{A_\infty} \right] \quad (11)$$

simultaneously for all $k \geq 1$ and all branches $\mathbf{b}_i \in \mathcal{B}_k$.

Note that this implies what we have said previously: if the sequence $(n_k)_{k \geq 1}$ grows sufficiently fast, then the event $\{\mathcal{B}_k \neq \emptyset, \forall k \geq 1\}$ has a probability larger than $1 - \varepsilon$.

Proof. This is a direct application of Lemma 18. Imagine that n_1, \dots, n_k have been fixed and that \mathcal{B}_k has been constructed and is non empty. Fix $\mathbf{b}_i \in \mathcal{B}_k$. Using Lemma 18 (i) with $S = \mathbf{b}_i$, $n = n_{k+1}$ and $\eta = 2^{-k}$, we get

$$\begin{aligned} \mathbb{P} \left(\left| |\mathcal{T}(\mathbf{b}_i) \cap \mathcal{B}_{k+1}| - \frac{a_i \ell_{n_{k+1}}}{A_\infty} \right| \geq 2^{-k} \frac{a_i \ell_{n_{k+1}}}{A_\infty} \right) &\leq 4^k (n_{k+1})^{-c+o(1)} / a_i \\ &\leq_{\mathbf{b}_i \text{ is good}} 4^k (n_{k+1})^{-c+o(1)} n_k^{\alpha+\varepsilon}. \end{aligned}$$

Given n_k , we can thus choose $f(n_k)$ large enough so that if $n_{k+1} \geq f(n_k)$ the right-hand side of the last display is smaller than $2^{-k}\varepsilon/(n_k + 1)$. For such an integer n_{k+1} , the probability that one of the branches \mathbf{b}_i of \mathcal{B}_k does not satisfy (11) is smaller than

$$(n_k + 1) \cdot 2^{-k}\varepsilon/(n_k + 1) = 2^{-k}\varepsilon.$$

Constructing in this way a sequence $(n_k)_{k \geq 1}$, we see that the probability that (11) fails for one k is smaller than $\varepsilon \cdot (2^{-1} + 2^{-2} + \dots) = \varepsilon$. \square

Lemma 20. *There exists a function $g : \mathbb{N} \rightarrow \mathbb{N}$ with $g(n) > 2n$ for all $n \geq 1$, such that if the sequence $(n_k)_{k \geq 1}$ satisfies $n_{k+1} \geq g(n_k)$ for all $k \geq 1$, then with probability at least $1 - \varepsilon$, for all $k \geq 1$ we have*

$$\sup_{x \in \mathcal{T}} \# \{ \mathbf{b}_i \in \mathcal{B}_k : \mathbf{b}_i \cap B(x, n_k^{-\alpha}) \neq \emptyset \} \leq n_k^\varepsilon$$

Proof. Imagine that \mathcal{B}_k is constructed and pick $\mathbf{b}_i \in \mathcal{B}_k$. Conditionally on the number N of branches of \mathcal{B}_{k+1} grafted onto \mathbf{b}_i , the grafting points of these branches are i.i.d. and uniform on \mathbf{b}_i . We decompose the good branch \mathbf{b}_i into $\lceil a_i/n_{k+1}^{-\alpha} \rceil$ intervals of length smaller than $n_{k+1}^{-\alpha}$. If none of these intervals contains more than $n_{k+1}^{\varepsilon/2}$ branches then it is not possible to have more than $3n_{k+1}^{\varepsilon/2}$ branches within distance less than $n_{k+1}^{-\alpha}$. Noticing that $N \leq n_{k+1} + 1$, we get that the probability to have more than $3n_{k+1}^{\varepsilon/2}$ branches within distance less than $n_{k+1}^{-\alpha}$ is smaller than

$$\begin{aligned} \left\lceil \frac{a_i}{n_{k+1}^{-\alpha}} \right\rceil \cdot \binom{N}{n_{k+1}^{\varepsilon/2}} \left(\frac{n_{k+1}^{-\alpha}}{a_i} \right)^{n_{k+1}^{\varepsilon/2}} &\leq \left(\frac{n_{k+1}^{-\alpha+o(1)}}{n_{k+1}^{-\alpha}} + 1 \right) \cdot (n_{k+1} + 1)^{n_{k+1}^{\varepsilon/2}} n_{k+1}^{-\alpha \cdot n_{k+1}^{\varepsilon/2}} n_k^{(\alpha+\varepsilon) \cdot n_{k+1}^{\varepsilon/2}} \\ &\leq (n_{k+1}^{-\alpha+1+o(1)} n_k^{\alpha+\varepsilon/2+o(1)})^{n_{k+1}^{\varepsilon/2}}. \end{aligned}$$

Clearly by making $n_{k+1} \geq g(n_k)$ grows rapidly enough we can ensure that the series of the last probabilities is as small as we wish. Hence with probability at least $1 - \varepsilon$, for every $k \geq 2$ and any $x \in \mathcal{T}$, the number of branches of \mathcal{B}_k grafted on a given $\mathbf{b}_i \in \mathcal{B}_{k-1}$ within distance $n_k^{-\alpha}$ of x is at most $3n_k^{\varepsilon/2}$. Using this proposition in cascades (and remarking that $n_i^{-\alpha} > n_k^{-\alpha}$ for $i < k$), we get that on this event

$$\sup_{x \in \mathcal{T}} \# \{ \mathbf{b}_i \in \mathcal{B}_k : \mathbf{b}_i \cap B(x, n_k^{-\alpha}) \neq \emptyset \} \leq 3n_1^{\varepsilon/2} \dots 3n_{k-1}^{\varepsilon/2} 3n_k^{\varepsilon/2},$$

and the last product is less than n_k^ε provided that n_k grows rapidly enough. \square

We will now use this lemma and Lemma 18 to control the maximal length of groups of branches of \mathcal{B}_{k+1} that are grafted on a ball of radius r , when the center of the ball runs over \mathcal{B}_k . In that aim, we also need to assume that the sequence (n_k) grows sufficiently fast so that

$$n_k = n_{k+1}^{o(1)} \quad \text{as } k \rightarrow \infty. \quad (12)$$

Corollary 21. *Assume that the sequence (n_k) satisfies $n_{k+1} \geq g(n_k)$ for all k – where g is the function of the previous lemma – as well as (12). For each $k \in \mathbb{N}$, each $r > 0$ and each $x \in \mathcal{B}_k$, consider the total length of branches of \mathcal{B}_{k+1} that are grafted on $B(x, r) \cap \mathcal{B}_k \subset \mathcal{T}_{n_{k+1}-1}$. Let $\mathcal{L}_{k+1}(r)$ be the supremum of these lengths when x runs over \mathcal{B}_k . Then with probability at least $1 - \varepsilon$, for all $0 < \gamma < 1 - \varepsilon/\alpha$ and for all k large enough (the threshold depending on γ),*

$$\mathcal{L}_{k+1}(r) \leq r^{\frac{1}{\alpha} - \varepsilon} \ell_{n_{k+1}} \quad \text{for all } r \in \left[n_{k+1}^{-\alpha}, n_{k+1}^{-1+\frac{\varepsilon}{2}} \right]$$

and

$$\mathcal{L}_{k+1}(r) \leq r^\gamma \ell_{n_{k+1}} \quad \text{for all } r \in \left[n_{k+1}^{-1+\frac{\varepsilon}{2}}, n_k^{-\alpha} \right].$$

Proof. Let \mathcal{A} denote the event of probability at least $1 - \varepsilon$ on which the conclusion of Lemma 20 holds. In the following, we will work mostly on \mathcal{A} and $\gamma \in (0, 1 - \varepsilon/\alpha)$ is fixed.

To start with, we set up for each $r \in [n_{k+1}^{-\alpha}, n_k^{-\alpha}]$ a specific covering of \mathcal{B}_k . Split each $\mathbf{b}_i \in \mathcal{B}_k$ into $\lceil a_i/r \rceil$ intervals, with $\lfloor a_i/r \rfloor$ intervals of length r and a last one (if a_i/r is not an integer) of length smaller than r which is chosen to be the one that reaches the leaf of \mathbf{b}_i . This gives a set of

$$\sum_{i: \mathbf{b}_i \in \mathcal{B}_k} \left\lceil \frac{a_i}{r} \right\rceil \leq \frac{|\mathcal{B}_k|}{r} + \#G_{n_k} \leq \frac{A_\infty}{r} + n_k + 1$$

intervals of \mathcal{B}_k of lengths smaller than r . Besides, consider the balls of radius r centered at the points of $\mathcal{B}_{k-1} \cap \mathcal{B}_k$ (i.e. at the “roots” of the $\mathbf{b}_i, \mathbf{b}_i \in \mathcal{B}_k$). For such a ball B , the set $B \cap \mathcal{B}_k$ intersects at most n_k^ε branches $\mathbf{b}_i, \mathbf{b}_i \in \mathcal{B}_k$, conditionally on \mathcal{A} (by Lemma 20). In particular, its length $|B \cap \mathcal{B}_k|$ is smaller than $n_k^\varepsilon r$. The covering we are interested in is composed by the intersections of these balls with \mathcal{B}_k and the intervals mentioned above. It is therefore composed by sets that all have a length smaller than $n_k^\varepsilon r$. Moreover, each ball of radius r centered at a point of \mathcal{B}_k is included in the union of two neighboring elements of the covering, one of which being necessarily an interval.

• Using this covering, we note that

$$\begin{aligned} & \mathbb{P} \left(\exists r \in \left[n_{k+1}^{-\alpha}, n_{k+1}^{-1+\frac{\varepsilon}{2}} \right] : \mathcal{L}_{k+1}(r) \geq r^{\frac{1}{\alpha}-\varepsilon} \ell_{n_{k+1}}, \mathcal{A} \right) \\ & \leq \mathbb{P} \left(\mathcal{L}_{k+1}(n_{k+1}^{-1+\frac{\varepsilon}{2}}) \geq (n_{k+1}^{-\alpha})^{\frac{1}{\alpha}-\varepsilon} \ell_{n_{k+1}}, \mathcal{A} \right) \\ & \leq \left(A_\infty n_{k+1}^{1-\frac{\varepsilon}{2}} + n_k + 1 \right) \cdot 2\mathbb{P} \left(\mathcal{X} \geq 2^{-1} n_{k+1}^{-1+\alpha\varepsilon} \ell_{n_{k+1}} \right), \end{aligned}$$

where \mathcal{X} represents the total length of branches of \mathcal{B}_{k+1} that are grafted on a subset $S \subset \mathcal{T}_{n_{k+1}-1}$ of length $n_k^\varepsilon n_{k+1}^{-1+\varepsilon/2}$. By Lemma 18 (ii), for all integers $m \geq 1$ and then all k large enough, we have

$$\begin{aligned} \mathbb{P} \left(\exists r \in \left[n_{k+1}^{-\alpha}, n_{k+1}^{-1+\frac{\varepsilon}{2}} \right] : \mathcal{L}_{k+1}(r) \geq r^{\frac{1}{\alpha}-\varepsilon} \ell_{n_{k+1}}, \mathcal{A} \right) & \leq C'_m \left(A_\infty n_{k+1}^{1-\frac{\varepsilon}{2}} + n_k + 1 \right) \frac{n_k^{\varepsilon m} n_{k+1}^{(-1+\varepsilon/2)m} \ell_{n_{k+1}}^m}{n_{k+1}^{(-1+\alpha\varepsilon)m} \ell_{n_{k+1}}^m} \\ & \leq n_{k+1}^{1-\frac{\varepsilon}{2} + (\frac{1}{2}-\alpha)\varepsilon m + o(1)}. \end{aligned}$$

Fix m large enough so that the exponent $1 - \varepsilon/2 + (1/2 - \alpha)\varepsilon m \leq -1$. Since $n_{k+1} \geq 2^k$ for all k , we can therefore use Borel-Cantelli's lemma to conclude that on \mathcal{A} , almost surely for all k large enough,

$$\mathcal{L}_{k+1}(r) \leq r^{\frac{1}{\alpha}-\varepsilon} \ell_{n_{k+1}} \quad \text{for all } r \in \left[n_{k+1}^{-\alpha}, n_{k+1}^{-1+\frac{\varepsilon}{2}} \right].$$

• For $r \in [n_{k+1}^{-1+\varepsilon/2}, n_k^{-\alpha}]$ the argument is similar but we have to split the interval $[n_{k+1}^{-1+\varepsilon/2}, n_k^{-\alpha}]$ into subintervals to conclude. Let $\eta \in (1, (1 - \varepsilon\alpha^{-1})/\gamma)$ and first note that

$$\begin{aligned} \mathbb{P} \left(\exists r \in \left[n_{k+1}^{-1+\frac{\varepsilon}{2}}, n_k^{-\alpha} \right] : \mathcal{L}_{k+1}(r) \geq r^\gamma \ell_{n_{k+1}}, \mathcal{A} \right) & \leq \sum_{n=0}^{N_k} \mathbb{P} \left(\exists r \in \left[n_k^{-\alpha\eta^{n+1}}, n_k^{-\alpha\eta^n} \right] : \mathcal{L}_{k+1}(r) \geq r^\gamma \ell_{n_{k+1}}, \mathcal{A} \right) \\ & \leq \sum_{n=0}^{N_k} \mathbb{P} \left(\mathcal{L}_{k+1}(n_k^{-\alpha\eta^n}) \geq n_k^{-\alpha\gamma\eta^{n+1}} \ell_{n_{k+1}}, \mathcal{A} \right), \end{aligned}$$

where N_k is the largest integer n such that $n_k^{-\alpha\eta^n} \geq n_{k+1}^{-1+\varepsilon/2}$. Applying Lemma 18 (ii) to subsets S of $\mathcal{T}_{n_{k+1}-1}$ of lengths $n_k^\varepsilon n_k^{-\alpha\eta^n}$, we see that for all integers $m \geq 1$ and then all k large enough and all $n \leq N_k$,

$$\begin{aligned} \mathbb{P} \left(\mathcal{L}_{k+1}(n_k^{-\alpha\eta^n}) \geq n_k^{-\alpha\gamma\eta^{n+1}} \ell_{n_{k+1}}, \mathcal{A} \right) & \leq C_m \left(A_\infty n_k^{\alpha\eta^n} + n_k + 1 \right) \frac{(n_k^\varepsilon n_k^{-\alpha\eta^n})^m \ell_{n_{k+1}}^m}{(n_k^{-\alpha\gamma\eta^{n+1}})^m \ell_{n_{k+1}}^m} \\ & \leq C'_m n_k^{(\alpha + (\varepsilon + \alpha(\gamma\eta - 1))m)\eta^n}, \end{aligned}$$

where we have used for the last inequality that $\eta^n \geq 1$ and $\alpha > 1$. The parameters have been chosen so that $\varepsilon + \alpha(\gamma\eta - 1) < 0$. So we can fix m sufficiently large so that $\alpha + (\varepsilon + \alpha(\gamma\eta - 1))m \leq -1$ and then conclude that for all k large enough

$$\begin{aligned} \mathbb{P}\left(\exists r \in \left[n_{k+1}^{-1+\frac{\varepsilon}{2}}, n_k^{-\alpha}\right] : \mathcal{L}_{k+1}(r) \geq r^\gamma \ell_{n_{k+1}}, A\right) &\leq C'_m \sum_{n=0}^{N_k} \frac{1}{n_k^{\eta^n}} \\ &\stackrel{n_k \geq 2^{k-1}}{\leq} \frac{C'_m}{2^{(k-1)}} \sum_{n=0}^{\infty} \frac{1}{2^{(k-1)(\eta^n - 1)}} \\ &\leq \frac{C'_m}{2^{(k-1)}} \sum_{n=0}^{\infty} \frac{1}{2^{\eta^n - 1}} \end{aligned}$$

and the series, clearly, is convergent. Again, we conclude with Borel-Cantelli's lemma that a.s. on \mathcal{A} , for all k large enough,

$$\mathcal{L}_{k+1}(r) \leq r^\gamma \ell_{n_{k+1}} \quad \text{for all } r \in \left[n_{k+1}^{-1+\frac{\varepsilon}{2}}, n_k^{-\alpha}\right].$$

□

3.2.2 Proof of Proposition 16

Fix $\gamma \in (1 - \varepsilon, 1 - \varepsilon/\alpha)$ and fix a sequence $(n_k)_{k \geq 1}$ such that the conditions of Corollary 19 and Corollary 21 are satisfied (in particular (12) holds). There exists therefore an event \mathcal{E} of probability at least $1 - 2\varepsilon$ on which the conclusions of Lemma 17, Corollary 19 and Corollary 21 hold, for the γ we have chosen. From now on, we work on this event \mathcal{E} and it is implicit in what follows that all assertions hold conditionally on \mathcal{E} . By Corollary 19, each branch of \mathcal{B}_k will have some branches of \mathcal{B}_{k+1} grafted on it and so $\mathcal{B}_k \neq \emptyset$ for all $k \geq 1$ and the measures π_k are well-defined for all $k \geq 1$. We denote by π a subsequential limit of (π_k) . We aim at proving (9).

By Corollary 19 again, for all $k \geq 1$

$$|\mathcal{B}_{k+1}| \in \left[(1 - 2^{-k}) \frac{|\mathcal{B}_k| \ell_{n_{k+1}}}{A_\infty}, (1 + 2^{-k}) \frac{|\mathcal{B}_k| \ell_{n_{k+1}}}{A_\infty}\right]. \quad (13)$$

Consequently,

$$|\mathcal{B}_{k+1}| \stackrel{(10)}{=} n_{k+1}^{1-\alpha+o(1)} |\mathcal{B}_k| \stackrel{(12)}{=} n_{k+1}^{1-\alpha+o(1)}. \quad (14)$$

Next, using Corollary 19 as well as (13) in cascades, we see that for any $\mathbf{b}_i \in \mathcal{B}_k$ and any $k' \geq k$

$$\begin{aligned} |\mathcal{T}(\mathbf{b}_i) \cap \mathcal{B}_{k'}| &\in a_i \cdot \left[\prod_{j=k+1}^{k'} (1 - 2^{-(j-1)}) \frac{\ell_{n_j}}{A_\infty}; \prod_{j=k+1}^{k'} (1 + 2^{-(j-1)}) \frac{\ell_{n_j}}{A_\infty} \right]. \\ |\mathcal{B}_{k'}| &\in |\mathcal{B}_k| \cdot \left[\prod_{j=k+1}^{k'} (1 - 2^{-(j-1)}) \frac{\ell_{n_j}}{A_\infty}; \prod_{j=k+1}^{k'} (1 + 2^{-(j-1)}) \frac{\ell_{n_j}}{A_\infty} \right]. \end{aligned}$$

Let $c_1 = \prod_{j=1}^{\infty} (1 - 2^{-j}) / (1 + 2^{-j}) \in (0, \infty)$ and $c_2 = \prod_{j=1}^{\infty} (1 + 2^{-j}) / (1 - 2^{-j}) \in (0, \infty)$, then we have

$$\pi_{k'}(\mathcal{T}(\mathbf{b}_i)) = \frac{|\mathcal{T}(\mathbf{b}_i) \cap \mathcal{B}_{k'}|}{|\mathcal{B}_{k'}|} \in \frac{a_i}{|\mathcal{B}_k|} \cdot [c_1, c_2].$$

Using arguments similar to those developed in the proof of Lemma 17 we get that for any branch $\mathbf{b}_i \in \mathcal{B}_k$

$$\pi(\mathcal{T}(\mathbf{b}_i)) \in \left[\frac{c_1}{c_2} \frac{a_i}{|\mathcal{B}_k|}, \frac{c_2}{c_1} \frac{a_i}{|\mathcal{B}_k|} \right]. \quad (15)$$

Now, recall that the support of the measure π is included in $\cap_{k \geq 1} \mathcal{T}(\mathcal{B}_k)$ (by Lemma 17) and fix $x \in \cap_{k \geq 1} \mathcal{T}(\mathcal{B}_k)$. Let $r \in [n_{k+1}^{-\alpha}, n_k^{-\alpha}]$ for some $k \in \mathbb{N}$ and note that

$$\begin{aligned} \pi(B(x, r)) &= \sum_{i: \mathbf{b}_i \in \mathcal{B}_{k+1}} \pi(B(x, r) \cap \mathcal{T}(\mathbf{b}_i)) \\ &\stackrel{(15)}{\leq} \frac{c_2}{c_1 |\mathcal{B}_{k+1}|} \sum_{i: \mathbf{b}_i \in \mathcal{B}_{k+1}} a_i \mathbb{1}_{\{B(x, r) \cap \mathcal{T}(\mathbf{b}_i) \neq \emptyset\}}. \end{aligned}$$

Note also that $\sum_{i: \mathbf{b}_i \in \mathcal{B}_{k+1}} a_i \mathbb{1}_{\{B(x, r) \cap \mathcal{T}(\mathbf{b}_i) \neq \emptyset\}} \leq \mathcal{L}_{k+1}(r)$, with the notation of Corollary 21. (The bounds below will therefore be true simultaneously for all x .) Hence, according to this corollary,

$$\pi(B(x, r)) \leq \frac{c_2 \ell_{n_{k+1}} r^{\frac{1}{\alpha} - \varepsilon}}{c_1 |\mathcal{B}_{k+1}|} \stackrel{(13)}{\leq} \frac{c_2 A_\infty}{c_1 (1 - 2^{-k})} \cdot \frac{r^{\frac{1}{\alpha} - \varepsilon}}{|\mathcal{B}_k|} \leq r^{1/\alpha - 2\varepsilon} \quad \text{for all } r \in [n_{k+1}^{-\alpha}, n_k^{-1 + \frac{\varepsilon}{2}}]$$

provided that k is large enough, since $|\mathcal{B}_k| = n_k^{1 - \alpha + o(1)} = n_{k+1}^{o(1)}$, by (14) and (12). On the other hand, again by Corollary 21,

$$\pi(B(x, r)) \leq \frac{c_2 A_\infty}{c_1 (1 - 2^{-k})} \cdot \frac{r^\gamma}{|\mathcal{B}_k|} \stackrel{(14)}{=} \frac{r^\gamma}{n_k^{1 - \alpha + o(1)}} \quad \text{for all } r \in [n_{k+1}^{-1 + \frac{\varepsilon}{2}}, n_k^{-\alpha}],$$

where the $o(1)$ is independent of r . Recall that $\gamma > 1 - \varepsilon$ and then note that $r \leq n_k^{-\alpha}$ implies $r^{\gamma - 1/\alpha + \varepsilon} \leq n_k^{1 - \alpha\gamma - \alpha\varepsilon}$, hence $r^\gamma n_k^{-1 + \alpha + o(1)} \leq r^{1/\alpha - \varepsilon}$ for all k large enough (independently of r).

In conclusion, on the event \mathcal{E} , for all k large enough and then all $r \in [n_{k+1}^{-\alpha}, n_k^{-\alpha}]$ – hence for all r sufficiently small,

$$\pi(B(x, r)) \leq r^{1/\alpha - 2\varepsilon} \quad \text{for all } x \in \bigcap_{k \geq 1} \mathcal{T}(\mathcal{B}_k),$$

which implies (9) since the support of π is included in $\cap_{k \geq 1} \mathcal{T}(\mathcal{B}_k)$ by Lemma 17.

4 Appendix

We gather here some elementary technical results useful in the core of the paper. Let $(a_i, i \geq 1)$ be a sequence of strictly positive real numbers, and $A_i = a_1 + \dots + a_i$, $i \geq 1$.

Lemma 22. (i) *The series $\sum_i \frac{a_i}{A_i^2}$ and $\sum_i \left(\frac{a_i}{A_i}\right)^2$ are convergent.*

(ii) *If $a_i \leq i^{-\alpha + o(1)}$ for some $\alpha > 0$, then $\sum_{i \geq n} \frac{a_i^2}{A_i} \leq n^{-\alpha + o(1)}$.*

(iii) *If $A_i \geq i^{1 - \alpha + o(1)}$ for some $\alpha \in (0, 1)$, then $\sum_{i \geq n} \frac{a_i}{A_i^2} \leq n^{\alpha - 1 + o(1)}$.*

Proof. Since the sequence (A_i^{-1}) is bounded from above, Assertions (i) and (ii) are immediate when the series $\sum_i a_i$ is convergent. (Assertion (iii) requires anyway that the series $\sum_i a_i$ is divergent.) So we assume from now on that the series $\sum_i a_i$ diverges, and define for all $k \geq 1$

$$n_k := \inf\{i \geq 1 : A_i \geq k\},$$

which is finite. Note that $A_{n_k} \geq k$ and $A_{n_{k+1}-1} < k+1$, in particular $A_{n_{k+1}-1} - A_{n_k} < 1$.

Assertion (i). The convergence of the series $\sum_i \frac{a_i}{A_i^2}$ is simply due to the following observation :

$$\sum_{i=n_1}^{\infty} \frac{a_i}{A_i^2} = \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_{k+1}-1} \frac{a_i}{A_i^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{i=n_k}^{n_{k+1}-1} a_i < \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

The proof is similar for the series $\sum_i \left(\frac{a_i}{A_i}\right)^2$, since $\sum_{i=n_k}^{n_{k+1}-1} a_i^2 \leq (\sum_{i=n_k}^{n_{k+1}-1} a_i)^2 < 1$.

Assertion (ii). We assume that $a_i \leq i^{-\alpha+o(1)}$ for some $\alpha \in (0, 1]$. Let $\varepsilon \in (0, \alpha/2)$. For i large enough, we have $A_i \leq i^{1-\alpha+\varepsilon}$ and therefore, for k large enough, $n_k \geq k^{1/(1-\alpha+\varepsilon)}$. Consequently, for all $i \geq \max(n, n_k)$, with n and k large enough,

$$a_i \leq i^{-\alpha+\varepsilon} = i^{-\alpha+2\varepsilon} \times i^{-\varepsilon} \leq n^{-\alpha+2\varepsilon} \times k^{-\varepsilon/(1-\alpha+\varepsilon)}.$$

And then, for n large enough,

$$\begin{aligned} \sum_{i \geq n} \frac{a_i^2}{A_i} &= \sum_{k \geq 1} \sum_{i=n_k}^{n_{k+1}-1} \mathbb{1}_{\{i \geq n\}} \frac{a_i^2}{A_i} \\ &\leq n^{-\alpha+2\varepsilon} \sum_{k \geq 1} \frac{k^{-\varepsilon/(1-\alpha+\varepsilon)}}{k} \left(\sum_{i=n_k}^{n_{k+1}-1} a_i \right) \\ &\leq n^{-\alpha+2\varepsilon} \sum_{k \geq 1} \frac{1}{k^{1+\varepsilon/(1-\alpha+\varepsilon)}}. \end{aligned}$$

This holds for all $\varepsilon > 0$ small enough and the conclusion follows.

Assertion (iii). Fix $\varepsilon \in (0, (1-\alpha)/2)$. For i large enough, $A_i \geq i^{1-\alpha-\varepsilon}$. Hence for $i \geq \max(n, n_k)$, with n large enough,

$$A_i^2 \geq A_n^{1-\varepsilon} A_{n_k}^{1+\varepsilon} \geq n^{1-\alpha-2\varepsilon} k^{1+\varepsilon}.$$

Consequently, for n large enough

$$\sum_{i \geq n} \frac{a_i}{A_i^2} = \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_{k+1}-1} \mathbb{1}_{\{i \geq n\}} \frac{a_i}{A_i^2} \leq n^{\alpha-1+2\varepsilon} \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}}.$$

□

References

- [1] D. ALDOUS, *The continuum random tree. I*, Ann. Probab., 19 (1991), pp. 1–28.
- [2] O. AMINI, L. DEVROYE, S. GRIFFITHS, AND N. OLVER, *Explosion and linear transit times in infinite trees*, (2014).
- [3] M. T. BARLOW, R. PEMANTLE, AND E. A. PERKINS, *Diffusion-limited aggregation on a tree*, Probab. Theory Related Fields, 107 (1997), pp. 1–60.
- [4] S. N. EVANS, *Probability and real trees*, vol. 1920 of Lecture Notes in Mathematics, Springer, Berlin, 2008. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005.
- [5] K. FALCONER, *Fractal geometry*, John Wiley & Sons, Inc., Hoboken, NJ, second ed., 2003. Mathematical foundations and applications.
- [6] C. GOLDSCHMIDT AND B. HAAS, *A line-breaking construction of the stable trees*, (2014).
- [7] W. IMRICH, *On metric properties of tree-like spaces*, in Contributions to graph theory and its applications (Internat. Colloq., Oberhof, 1977) (German), Tech. Hochschule Ilmenau, Ilmenau, 1977, pp. 129–156.
- [8] J.-F. LE GALL, *Random real trees*, Ann. Fac. Sci. Toulouse Math. (6), 15 (2006), pp. 35–62.
- [9] R. PEMANTLE, *A time-dependent version of Pólya’s urn*, J. Theoret. Probab., 3 (1990), pp. 627–637.
- [10] O. SCHRAMM, *Conformally invariant scaling limits : an overview and a collection of problems*, Plenary Lecture ICM Madrid 2006, (2006).